

GEOMETRIC SCHUR DUALITY OF CLASSICAL TYPE

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ABSTRACT. This is a generalization of the classic work of Beilinson, Lusztig and MacPherson. In this paper (and an Appendix) we show that the quantum algebras obtained via a BLM-type stabilization procedure in the setting of partial flag varieties of type B/C are two (modified) coideal subalgebras of the quantum general linear Lie algebra, $\dot{\mathbf{U}}^J$ and $\dot{\mathbf{U}}^i$. We provide a geometric realization of the Schur-type duality of Bao-Wang between such a coideal algebra and Iwahori-Hecke algebra of type B . The monomial bases and canonical bases of the Schur algebras and the modified coideal algebra $\dot{\mathbf{U}}^J$ are constructed.

In an Appendix by three authors, a more subtle 2-step stabilization procedure leading to $\dot{\mathbf{U}}^i$ is developed, and then monomial and canonical bases of $\dot{\mathbf{U}}^i$ are constructed. It is shown that $\dot{\mathbf{U}}^i$ is a subquotient of $\dot{\mathbf{U}}^J$ with compatible canonical bases. Moreover, a compatibility between canonical bases for modified coideal algebras and Schur algebras is established.

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1. INTRODUCTION

1.1. In an influential 1990 paper [BLM], Beilinson, Lusztig and MacPherson (BLM) provided a geometric construction of the quantum Schur algebra and more importantly the (modified) quantum groups $\mathbf{U}(\mathfrak{gl}(N))$ and $\dot{\mathbf{U}}(\mathfrak{gl}(N))$, using the partial flag varieties of type A . They further constructed a canonical basis for this modified quantum group (see Lusztig [Lu93] for generalization to all types). The BLM construction has played a fundamental role in categorification; see Khovanov-Lauda [KhL10]. The BLM construction is adapted subsequently by Grojnowski and Lusztig [GL92] to realize geometrically the Schur-Jimbo duality [Jim86]. These works raised an immediate question which remains open until now:

(Q1). What are the quantum algebras arising from flag varieties of classical type?

At the beginning, one was even tempted to hope that the corresponding Drinfeld-Jimbo quantum groups [Dr86] would provide the answer. However, the expectation for an answer being the quantum groups of classical type was somewhat diminished after Nakajima's quiver variety construction [Na94] which has provided a powerful geometric realization of integrable modules of (quantum) Kac-Moody algebras of symmetric type. Nevertheless, there has been a successful generalization to the affine type A in [GV93, Lu99, Mc12, VV99, SV00] via affine flag varieties of type A .

In a seemingly unrelated direction two of the authors [BW13] recently developed a new approach to Kazhdan-Lusztig theory for the BGG category \mathcal{O} of classical type by initiating a new theory of *canonical basis* arising from quantum symmetric pairs, and used it to solve the irreducible character problem for the ortho-symplectic Lie superalgebras. At a decategorification level, a duality was established in *loc. cit.* between a quantum algebra (denoted by \mathbf{U}^i or \mathbf{U}^j) and the Iwahori-Hecke algebra \mathbf{H}_{B_d} of type B acting on a tensor space, generalizing the Schur-Jimbo duality (we shall refer to this new duality as *\imath Schur duality*, where *\imath* partly stands for “involution”). The *\imath Schur duality* can also be formulated

in terms of Schur algebras \mathbf{S}^i or \mathbf{S}^j , and indeed an algebraic version of the $(\mathbf{S}^i, \mathbf{H}_{B_d})$ -duality already appeared in [G97].

The aforementioned quantum algebras \mathbf{U}^j and \mathbf{U}^i are the so-called coideal subalgebras of the quantum group $\mathbf{U}(\mathfrak{gl}(N))$ and they form quantum symmetric pairs $(\mathbf{U}(\mathfrak{gl}(N)), \mathbf{U}^i)$ and $(\mathbf{U}(\mathfrak{gl}(N)), \mathbf{U}^j)$, depending on whether N is odd or even. A general theory of quantum symmetric pairs was developed by Letzter [Le02]. The categorical significance of i Schur duality [BW13] raises the following natural question:

(Q2). Is there a geometric realization of i Schur duality and i canonical basis?

1.2. The goal of this paper and an Appendix is to settle the two questions (Q1) and (Q2) for type B/C completely by showing they provide answers to each other.

The coideal algebras admit modified (i.e., idempotentized) versions $\dot{\mathbf{U}}^j$ and $\dot{\mathbf{U}}^i$, following the by-now-standard algebraic construction. In the paper we establish multiplication formulas, generating sets as well as canonical bases for the Schur algebras \mathbf{S}^j and \mathbf{S}^i via a geometric approach. We show that $\dot{\mathbf{U}}^j$ is the quantum algebra à la BLM stabilization behind the family of Schur algebras \mathbf{S}^j using the geometry of partial flag varieties of type B/C . As applications, the monomial and i canonical bases of the algebra $\dot{\mathbf{U}}^j$ are constructed for the first time, and the i Schur dualities are realized geometrically à la Grojnowski-Lusztig.

The Appendix written by three of the authors provides the more subtle 2-step stabilization procedure which leads to the remaining algebra $\dot{\mathbf{U}}^i$ and its canonical basis. Precise relations between $\dot{\mathbf{U}}^i$ and $\dot{\mathbf{U}}^j$ are established. It is also shown that there exist surjective homomorphisms from the modified coideal algebras to the Schur algebras (of both types), which send canonical bases to canonical bases or zero.

1.3. Let us describe our constructions and results in detail.

We provide in Section 2 a geometric convolution construction of a Schur \mathcal{A} -algebra \mathbf{S}^j on pairs of N -step partial flags of type B_d as a convolution algebra, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. This algebra and the Iwahori-Hecke algebra of type B_d (via a similar convolution construction on pairs of complete flags) satisfy a double centralizer property acting on an \mathcal{A} -module \mathbf{T}_d , which is also defined geometrically. We compute the geometric action of the Iwahori-Hecke algebra on \mathbf{T}_d explicitly.

In Section 3, various basic multiplication formulas for \mathbf{S}^j , \mathbf{H}_{B_d} , and their commuting actions on \mathbf{T}_d are worked out precisely (whose type A counterparts can be found in [BLM, GL92]). We establish a generating set for \mathbf{S}^j and several explicit relations satisfied by these generators. From geometry, we construct a standard basis, a monomial basis, and a canonical basis of \mathbf{S}^j , respectively. The signed canonical basis of \mathbf{S}^j is shown to be characterized by an almost orthonormality similar to the more familiar Lusztig-Kashiwara canonical basis [Lu90, Ka91, Lu93] arising from the Drinfeld-Jimbo quantum groups.

In Section 4, we establish a remarkable stabilization property for \mathbf{S}^j as d goes to infinity in a suitable sense, following the original approach of [BLM] (see [DDPW] for an exposition). This stabilization allows us to construct an \mathcal{A} -algebra \mathbf{K}^j . The algebra \mathbf{K}^j is again naturally equipped with a standard basis, a monomial basis, as well as a canonical basis. We give a presentation of the modified coideal algebra $\dot{\mathbf{U}}^j$, and establish an isomorphism of $\mathbb{Q}(v)$ -algebras $\dot{\mathbf{U}}^j \cong \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{K}^j$. We then establish a surjective \mathcal{A} -algebra homomorphism ϕ_d from \mathbf{K}^j to \mathbf{S}^j . We then identify the \mathbf{U}^j -module \mathbf{T}_d as a quantum d -th tensor space; in this

way, we obtain a geometric realization of the \imath Schur duality in [BW13] between \mathbf{U}^j and \mathbf{H}_{B_d} acting on \mathbf{T}_d .

So far we have been focused on the case of \mathbf{S}^j and $\dot{\mathbf{U}}^j$ in detail. One would like to have geometric realizations of the modified coideal algebra $\dot{\mathbf{U}}^i$ and related Schur algebra \mathbf{S}^i as well. As we shall see in Section 6, the Schur algebra \mathbf{S}^i arises most naturally using the type C flag varieties by constructions similar to Section 4. But in this way it is difficult if not impossible to see any direct connection between the two types of Schur algebras \mathbf{S}^i and \mathbf{S}^j (not to mention any further connection between $\dot{\mathbf{U}}^i$ and $\dot{\mathbf{U}}^j$), since we cannot compare the geometries of type B and C directly.

In Section 5 we provide geometric realizations, via a refined construction of type B_d partial flag varieties, of the Schur algebra \mathbf{S}^i , its canonical basis, as well as the \imath Schur $(\mathbf{S}^i, \mathbf{H}_{B_d})$ -duality. Among others, the geometric meaning of a distinguished generator \mathbf{t} in \mathbf{S}^i is made precise as sums of simple perverse sheaves, up to a shift. We further show that the standard bases, the monomial bases, as well as the canonical bases of \mathbf{S}^i and \mathbf{S}^j are all compatible under an algebra embedding $\mathbf{S}^i \subset \mathbf{S}^j$.

The constructions of the previous sections on Schur algebras are further adapted in the setting of flag varieties of type C in Section 6. It is interesting to note a Langlands type duality phenomenon, namely, \mathbf{S}^i arises most naturally in the type C setting as done in Section 4 (for type B where \mathbf{S}^j arises most naturally), and then \mathbf{S}^j appears also in a refined type C construction.

In Appendix A written by three of the authors, a more subtle stabilization procedure is developed to construct an \mathcal{A} -algebra \mathbf{K}^i from the family of algebras \mathbf{S}^i . The reason is that the divided power elements of the distinguished generator \mathbf{t} are poorly understood (see [BW13, Conjecture 4.13]) and there is no explicit multiplication formula in \mathbf{S}^i for these elements. Our construction of \mathbf{K}^i is a two-step process, and we show that \mathbf{K}^i is isomorphic to a subquotient of \mathbf{K}^j . It is further shown that $\dot{\mathbf{U}}^i \cong \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{K}^i$, and this leads to an integral \mathcal{A} -form ${}_{\mathcal{A}}\dot{\mathbf{U}}^i$. We then establish a remarkable property that the standard bases, the monomial bases, as well as the canonical bases are all compatible between \mathbf{K}^j and \mathbf{K}^i (equivalently, between ${}_{\mathcal{A}}\dot{\mathbf{U}}^j$ and ${}_{\mathcal{A}}\dot{\mathbf{U}}^i$) under the subquotient construction. We also construct a surjective homomorphism $\phi_d^i : \mathbf{K}^i \rightarrow \mathbf{S}^i$, and hence obtain a geometric realization of the $(\dot{\mathbf{U}}^i, \mathbf{H}_{B_d})$ -duality.

We further show in Appendix A that the surjective homomorphism $\phi_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$ in Section 4 maps an arbitrary canonical (respectively, standard) basis element of \mathbf{K}^j to a canonical (respectively, standard) basis element in \mathbf{S}^j or zero; see Theorem A.21 (compare [DF14]). An analogous compatibility of standard and canonical bases under the homomorphism $\phi_d^i : \mathbf{K}^i \rightarrow \mathbf{S}^i$ also holds.

We caution that the convention for \mathbf{U}^i , \mathbf{U}^j , and \mathbf{H}_{B_d} used in this paper are chosen to fit most naturally to the convention from geometry, and it is not quite the same as in [BW13].

1.4. This paper forms an essential part of a larger program as outlined in [BW13, §0.5]. There are a few followup projects of [BW13] and this paper which will be pursued elsewhere. The type D case will be treated separately. Various geometric realizations of coideal subalgebras of the quantum affine algebras of type A as well as classical symmetric pairs via Steinberg varieties will also be studied in depth. The constructions of \imath canonical bases with (partly conjectural) positivity are leading to an \imath categorification program, in which the geometric constructions in this paper will play a fundamental role.

Notations: \mathbb{N} denotes the set of nonnegative integers, $[a, b]$ denotes the set of integers between a and b .

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2. GEOMETRIC CONVOLUTION ALGEBRAS OF TYPE B

In this section, we construct a Schur algebra \mathbf{S}^j and Iwahori-Hecke algebra \mathbf{H}_{B_d} via convolution products in the framework of partial flag varieties of type B_d . We also construct a $(\mathbf{S}^j, \mathbf{H}_{B_d})$ -bimodule \mathbf{T}_d geometrically.

2.1. Preliminaries in type A . Let us fix a pair (N, D) of positive integers. Let \mathbb{F}_q be a finite field of q elements where q is always assumed to be odd in this paper. We shall denote by $|U|$ the dimension of a vector space U over \mathbb{F}_q . Consider the following data:

- The general linear group $GL(D)$ over \mathbb{F}_q of rank D ;
- The variety \tilde{X} of N -step flags $V = (0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{N-1} \subseteq V_N = \mathbb{F}_q^D)$ in \mathbb{F}_q^D ;
- The variety \tilde{Y} of complete flags $F = (0 = F_0 \subset F_1 \subset \cdots \subset F_{D-1} \subset F_D = \mathbb{F}_q^D)$ in \mathbb{F}_q^D .

Let $GL(D)$ act diagonally on the products $\tilde{X} \times \tilde{X}$, $\tilde{X} \times \tilde{Y}$ and $\tilde{Y} \times \tilde{Y}$. To a pair $(V, V') \in \tilde{X} \times \tilde{X}$, we can associate an $N \times N$ matrix $A = (a_{ij})$ by setting

$$a_{ij} = |(V_{i-1} + V_i \cap V'_j) / (V_{i-1} + V_i \cap V'_{j-1})|, \quad \forall 1 \leq i, j \leq N.$$

The above assignment $(V, V') \mapsto A$ defines a bijection

$$(2.1) \quad GL(D) \backslash \tilde{X} \times \tilde{X} \longleftrightarrow \Theta_D,$$

where $GL(D) \backslash \tilde{X} \times \tilde{X}$ is the set of $GL(D)$ -orbits in $\tilde{X} \times \tilde{X}$ and

$$\Theta_D = \left\{ A = (a_{ij}) \in \text{Mat}_{N \times N}(\mathbb{N}) \mid \sum_{i,j \in [1, N]} a_{ij} = D \right\}.$$

Here and below, $\text{Mat}_{k \times \ell}(R)$ denotes the set of $k \times \ell$ matrices with coefficients in R . Similar to (2.1), we have bijections

$$(2.2) \quad GL(D) \backslash \tilde{X} \times \tilde{Y} \longleftrightarrow \tilde{\Pi}, \quad GL(D) \backslash \tilde{Y} \times \tilde{Y} \longleftrightarrow \tilde{\Sigma},$$

where

$$\begin{aligned} \tilde{\Pi} &= \left\{ B = (b_{ij}) \in \text{Mat}_{N \times D}(\mathbb{N}) \mid \sum_{i \in [1, N]} b_{ij} = 1, \forall j \in [1, D] \right\}, \\ \tilde{\Sigma} &= \left\{ \sigma = (\sigma_{ij}) \in \text{Mat}_{D \times D}(\mathbb{N}) \mid \sum_{i \in [1, D]} \sigma_{ij} = 1 = \sum_{j \in [1, D]} \sigma_{ij}, \forall i, j \in [1, D] \right\}. \end{aligned}$$

By [BLM] and [GL92], we have

$$(2.3) \quad \#\tilde{\Sigma} = D!, \quad \#\tilde{\Pi} = N^D, \quad \text{and} \quad \#\Theta_D = \binom{N^2 + D - 1}{D}.$$

For any $N \times N$ matrix $A = (a_{ij})$, we define

$$\begin{aligned} \text{ro}(A) &= \left(\sum_j a_{1j}, \sum_j a_{2j}, \dots, \sum_j a_{Nj} \right), \\ \text{co}(A) &= \left(\sum_i a_{i1}, \sum_i a_{i2}, \dots, \sum_i a_{iN} \right). \end{aligned}$$

2.2. Parametrizing $O(D)$ -orbits. We fix a pair (n, d) of positive integers such that

$$N = 2n + 1, \quad D = 2d + 1,$$

where (N, D) is a pair of positive integers considered in Section 2.1.

Let us fix a non-degenerate symmetric bilinear form $Q : \mathbb{F}_q^D \times \mathbb{F}_q^D \rightarrow \mathbb{F}_q$. Let $O(D)$ be the orthogonal subgroup of $GL(D)$ consisting of elements g such that $Q(gu, gu') = Q(u, u')$ for any $u, u' \in \mathbb{F}_q^D$. If U is a subspace of \mathbb{F}_q^D , we write U^\perp for its orthogonal complement.

Consider the following subsets of \tilde{X} and \tilde{Y} :

- $X = \{V = (V_k) \in \tilde{X} \mid V_i = V_j^\perp, \text{ if } i + j = N\};$
- $Y = \{F = (F_\ell) \in \tilde{Y} \mid F_i = F_j^\perp, \text{ if } i + j = D\}.$

It is well known that $O(D)$ acts on X and Y . Let $O(D)$ act diagonally on $X \times X$, $X \times Y$ and $Y \times Y$, respectively. Consider the following subsets of Θ_D , $\tilde{\Pi}$ and $\tilde{\Sigma}$:

- $\Xi_d = \{A = (a_{ij}) \in \Theta_D \mid a_{ij} = a_{N+1-i, N+1-j}, \forall i, j \in [1, N]\};$
- $\Pi = \{B = (b_{ij}) \in \tilde{\Pi} \mid b_{ij} = b_{N+1-i, D+1-j}, \forall i \in [1, N], j \in [1, D]\};$
- $\Sigma = \{\sigma = (\sigma_{ij}) \in \tilde{\Sigma} \mid \sigma_{ij} = \sigma_{D+1-i, D+1-j}, \forall i, j \in [1, D]\}.$

Note that $a_{n+1, n+1}$ is odd for all $A \in \Xi_d$, and similarly, $b_{n+1, d+1} = 1$ for all $B \in \Pi$. Also note that ${}^t A \in \Xi_d$ for $A \in \Xi_d$, where ${}^t A$ denotes the transpose of A .

Lemma 2.1. *The bijections in (2.1) and (2.2) induce the following bijections:*

$$(2.4) \quad O(D) \backslash X \times X \longleftrightarrow \Xi_d, \quad O(D) \backslash X \times Y \longleftrightarrow \Pi, \quad \text{and} \quad O(D) \backslash Y \times Y \longleftrightarrow \Sigma.$$

(We shall denote the orbit corresponding to a matrix A by \mathcal{O}_A .)

Proof. The third bijection is the well-known Bruhat decomposition. We shall only prove the first one since the second one is similar.

Pick a pair $(V, V') \in X \times X$. Let A be the associated matrix under the bijection (2.1). We must show that $A \in \Xi_d$. Observe that we have

$$a_{ij} = |V_i \cap V_j'| - |V_{i-1} \cap V_j'| - |V_i \cap V_{j-1}'| + |V_{i-1} \cap V_{j-1}'|.$$

Since $V_i = V_{N-i}^\perp$, we have

$$\begin{aligned} |V_i \cap V_j'| &= |V_{N-i}^\perp \cap V_{N-j}'^\perp| = |(V_{N-i} + V_{N-j}')^\perp| \\ &= D - |V_{N-i}| - |V_{N-j}'| + |V_{N-i} \cap V_{N-j}'|. \end{aligned}$$

This implies that $a_{ij} = a_{N+1-i, N+1-j}$. So we have $A \in \Xi_d$.

Then we need to show that the map $(V, V') \mapsto A$ is surjective. If we take a pair $(F, F') \in Y \times Y$ and throw away F_i and F_j' for some fixed i and j , we get a pair $(V, V') \in X \times X$ (with the choice of $n = d - 1$). Suppose that σ is the associated matrix of (F, F') , it is clear from [BLM, 1.1] that the associated matrix A of (V, V') is the one obtained from σ by merging, i.e., adding componentwise, the i -th row (resp. $(D + 1 - i)$ -th) and $(i + 1)$ -th

row (resp. $(D - i)$ -th row) and then merging the j -th column (resp. $(D + 1 - j)$ -th) and $(j + 1)$ -th column (resp. $(D - j)$ -th column). So any matrix A in Ξ_d can be obtained from a not necessarily unique $\sigma \in \Sigma$ by repetitively applying the above observation. From this, we see that there is a pair $(V, V') \in X \times X$, obtained from a pair in $Y \times Y$ by throwing away subspaces at appropriate steps, such that its associated matrix is A . This shows that the map defined by $(V, V') \mapsto A$ is surjective.

We are left to show that the assignment of each $O(D)$ -orbit of (V, V') the matrix A is injective. Without loss of generality, we assume that $n \leq d$ and $V_i \neq V_j$ for any $i \neq j$. Under such assumption, we have a nonzero entry at each row and each column and each flag in X can be obtained from a flag in Y by dropping certain steps of flags. Let (V, V') and (\tilde{V}, V'') be two pairs in $X \times X$ such that their associated matrices are the same, say A . We further assume that all entries in A are either 0 or 1, except the entries (i_0, j_0) and $(N + 1 - i_0, N + 1 - j_0)$ at which A takes value 2. The general case can be shown in a similar argument. Since $O(D)$ acts transitively on X and Y , we can assume that $V = \tilde{V}$. We suppose that (V, V') and (V, V'') are obtained from two pairs (F, F') and (F, F'') in $Y \times Y$, respectively. It is clear then that the associated matrices (F, F') and (F, F'') differ only at a rank 2 submatrix whose upper left corner is (i_0, j_0) . If the associated matrix of (F, F') is one of them, the associated matrix of the pair (F, \tilde{F}') is the other, where \tilde{F}' is a flag such that $F'_i = \tilde{F}'_i$ for all $i \neq j_0, D + 1 - j_0$ and $F'_{i_0} \neq \tilde{F}'_{i_0}$. From this, we see that there is a g in the stabilizer of F in $O(D)$ such that $g(F'_i) = g(\tilde{F}'_i)$ for any $i \neq i_0, D + 1 - i_0$. This shows that (V, V') and (V, V'') are in the same $O(D)$ -orbit. We are done. \square

To each $B = (b_{i,j}) \in \Pi$, we associate a sequence $\mathbf{r} = (r_1, \dots, r_D)$ of integers where

$$(2.5) \quad r_c \text{ is the integer such that } b_{r_c, c} = 1 \text{ for each } c \in [1, D].$$

Note that \mathbf{r} is completely determined by the d -tuple $r_1 \cdots r_d$. This defines the following bijections

$$(2.6) \quad \begin{aligned} \Pi &\longleftrightarrow \{\text{sequences } \mathbf{r} = (r_1, \dots, r_D) \text{ with each } r_c \in [1, N] \text{ and } r_c + r_{D+1-c} = N + 1\} \\ &\longleftrightarrow \{\text{sequences } r_1 \cdots r_d \text{ with each } r_c \in [1, N]\}. \end{aligned}$$

Hence we can and shall denote the characteristic function of the $O(D)$ -orbit \mathcal{O}_B by $e_{r_1 \dots r_d}$.

The following is a counterpart of (2.3).

Lemma 2.2. *We have $\#\Sigma = 2^d \cdot d!$, $\#\Pi = (2n + 1)^d$, and $\#\Xi_d = \binom{2n^2 + 2n + d}{d}$.*

Proof. The first identity is well known and can be seen directly. The second identity follows from the bijection we defined before the lemma. We shall show the third identity.

Any matrix $A \in \Xi_d$ subject to the following condition:

$$\sum_{i,j \in [1, N]} a_{ij} = 2 \sum_{i \in [1, n], j \in [1, N]} a_{ij} + 2 \sum_{j \in [1, n]} a_{n+1, j} + a_{n+1, n+1} = 2d + 1.$$

So $a_{n+1, n+1}$ must be a positive odd number in $[1, 2d + 1]$ and

$$\sum_{i \in [1, n], j \in [1, N]} a_{ij} + \sum_{j \in [1, n]} a_{n+1, j} = \frac{2d + 1 - a_{n+1, n+1}}{2}.$$

Thus

$$\#\Xi_d = \sum_{l=0}^d \binom{2n^2 + 2n - 1 + d - l}{d - l} = \binom{2n^2 + 2n + d}{d}.$$

The lemma follows. \square

2.3. Convolution algebras in action. Let v be an indeterminate and $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. We define

$$\mathbf{S}^j = \mathcal{A}_{O(D)}(X \times X), \quad \mathbf{T}_d = \mathcal{A}_{O(D)}(X \times Y), \quad \mathbf{H}_{B_d} = \mathcal{A}_{O(D)}(Y \times Y)$$

to be the space of $O(D)$ -invariant \mathcal{A} -valued functions on $X \times X$, $X \times Y$, and $Y \times Y$ respectively. (Note by Lemma 2.1 that the parametrizations of the G -orbits are independent of the finite fields \mathbb{F}_q .) For $A \in \Xi_d$, we denote by e_A the characteristic function of the orbit \mathcal{O}_A . Then \mathbf{S}^j is a free \mathcal{A} -module with a basis $\{e_A \mid A \in \Xi_d\}$. Similarly, \mathbf{T}_d and \mathbf{H}_{B_d} are free \mathcal{A} -modules with bases parameterized by Π and Σ , respectively.

We define a convolution product $*$ on \mathbf{S}^j as follows. For a triple of matrices (A, A', A'') in $\Xi_d \times \Xi_d \times \Xi_d$, we choose $(f_1, f_2) \in \mathcal{O}_{A''}$, and we let $g_{A, A', A''; q}$ be the number of $f \in X$ such that $(f_1, f) \in \mathcal{O}_A$ and $(f, f_2) \in \mathcal{O}_{A'}$. A well-known property of the Iwahori-Hecke algebra implies that there exists a polynomial $g_{A, A', A''} \in \mathbb{Z}[v^2]$ such that $g_{A, A', A''; q} = g_{A, A', A''}|_{v=\sqrt{q}}$ for every odd prime power q . We define the convolution product on \mathbf{S}^j by letting

$$e_A * e_{A'} = \sum_{A''} g_{A, A', A''} e_{A''}.$$

Equipped with the convolution product, the \mathcal{A} -module \mathbf{S}^j becomes an associative \mathcal{A} -algebra. (A completely analogous convolution product gives us an \mathcal{A} -algebra structure on \mathbf{H}_{B_d} , which is well known to be the Iwahori-Hecke algebra of type B_d .)

An analogous convolution product (by regarding the triples (A, A', A'') as in $\Xi_d \times \Pi \times \Pi$ and $f \in Y$) gives us a left \mathbf{S}^j -action on \mathbf{T}_d ; a suitably modified convolution gives us a right \mathbf{H}_{B_d} -action on \mathbf{T}_d . These two actions commute and hence we have obtained an $(\mathbf{S}^j, \mathbf{H}_{B_d})$ -bimodule structure on \mathbf{T}_d . Denote

$${}_{\mathbb{Q}}\mathbf{S}^j = \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{S}^j, \quad {}_{\mathbb{Q}}\mathbf{H}_{B_d} = \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{H}_{B_d}, \quad {}_{\mathbb{Q}}\mathbf{T}_d = \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{T}_d.$$

Remark 2.3. Let us write X_n for the variety X of the N -step (isotropic) flags in \mathbb{F}_q^D and write ${}^n\mathbf{S}^j$ for \mathbf{S}^j for now (recall $N = 2n + 1$). Let $M = 2m + 1$ be another odd positive integer. Then via convolutions we can define a left action of ${}^n\mathbf{S}^j$ and a commuting right ${}^m\mathbf{S}^j$ -action on $\mathcal{A}_{O(D)}(X_n \times X_m)$ which can be shown to form double centralizers. This is a type B variant of the geometric symmetric Howe duality of the type A considered in [W01].

2.4. Geometric action of Iwahori-Hecke algebra. For $1 \leq j \leq d$, we define $T_j \in \mathbf{H}_{B_d}$ by

$$T_j(F, F') = \begin{cases} 1, & \text{if } F_i = F'_i \quad \forall i \in [1, d] \setminus \{j\} \text{ and } F_j \neq F'_j; \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that \mathbf{H}_{B_d} is isomorphic to the Iwahori-Hecke algebra of type B_d , which is an \mathcal{A} -algebra generated by T_i ($1 \leq i \leq d$) subject to the relations: $(T_i - v^2)(T_i + 1) = 0$ for $1 \leq i \leq d$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \leq i < d - 1$, $T_i T_j = T_j T_i$ for $|i - j| > 1$, $T_d T_{d-1} T_d T_{d-1} = T_{d-1} T_d T_{d-1} T_d$.

Lemma 2.4. *The right \mathbf{H}_{B_d} -action on \mathbf{T}_d ,*

$$\mathbf{T}_d \times \mathbf{H}_{B_d} \longrightarrow \mathbf{T}_d, \quad (e_{r_1 \dots r_d}, T_j) \mapsto e_{r_1 \dots r_d} T_j,$$

is given as follows. For $1 \leq j \leq d-1$, we have

$$(2.7) \quad e_{r_1 \dots r_d} T_j = \begin{cases} e_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j < r_{j+1}; \\ v^2 e_{r_1 \dots r_d}, & \text{if } r_j = r_{j+1}; \\ (v^2 - 1) e_{r_1 \dots r_d} + v^2 e_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j > r_{j+1}. \end{cases}$$

Moreover, (recalling $r_{d+2} = N + 1 - r_d$) we have

$$(2.8) \quad e_{r_1 \dots r_{d-1} r_d} T_d = \begin{cases} e_{r_1 \dots r_{d-1} r_{d+2}}, & \text{if } r_d < n + 1; \\ v^2 e_{r_1 \dots r_{d-1} r_d}, & \text{if } r_d = n + 1; \\ (v^2 - 1) e_{r_1 \dots r_{d-1} r_d} + v^2 e_{r_1 \dots r_{d-1} r_{d+2}}, & \text{if } r_d > n + 1. \end{cases}$$

Proof. It suffices to prove the formula for $v = \sqrt{q}$; by definition we interpret the convolution product over \mathbb{F}_q as $e_{r_1 \dots r_d} T_j(V, F) = \sum_{F' \in Y} e_{r_1 \dots r_d}(V, F') T_j(F', F)$.

The above formula (2.7) coincides with the one in [GL92, 1.12], whose proof is also the same as in the type A case. We shall prove (2.8). By definitions, we have

$$\begin{aligned} e_{r_1 \dots r_d} T_d(V, F) &= \sum_{F' \in Y} e_{r_1 \dots r_d}(V, F') T_d(F', F) \\ &= \sum_{F' \in Y: F'_d \neq F_d, F'_i = F_i, \forall i \in [1, d-1]} e_{r_1 \dots r_d}(V, F'). \end{aligned}$$

From the above expression, we see that the value of $e_{r_1 \dots r_d} T_d$ at (V, F) is zero if the associated sequence of (V, F) is not the one listed in the formula. It is also clear that the calculation is reduced to the case when $D = 3$ by comparing the pairs (V, F) with the pair obtained from (V, F) by intersecting with F_{d+2} and modulo F_{d-1} , and in this case the formula can be derived by a direct computation. (Note that the restriction of Q to the quotient F_{d+2}/F_{d-1} is again a non-degenerate form.) \square

We set

$$\tilde{e}_{r_1 \dots r_d} = v^{\#\{(c, c') \mid c, c' \in [1, d], c < c', r_c < r_{c'}\} + \epsilon} e_{r_1 \dots r_d},$$

where

$$\epsilon = \begin{cases} 1, & \text{if } r_d < n + 1; \\ 0, & \text{otherwise.} \end{cases}$$

The fomulas (2.7) and (2.8) can be rewritten as follows:

$$(2.9) \quad \tilde{e}_{r_1 \dots r_d} T_j = \begin{cases} v \tilde{e}_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j < r_{j+1}; \\ v^2 \tilde{e}_{r_1 \dots r_d}, & \text{if } r_j = r_{j+1}; \\ (v^2 - 1) \tilde{e}_{r_1 \dots r_d} + v \tilde{e}_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j > r_{j+1}, \end{cases}$$

for $1 \leq j \leq d-1$, and

$$(2.10) \quad \tilde{e}_{r_1 \dots r_d} T_d = \begin{cases} v \tilde{e}_{r_1 \dots r_{d-1} r_{d+2}}, & \text{if } r_d < n + 1; \\ v^2 \tilde{e}_{r_1 \dots r_d}, & \text{if } r_d = n + 1; \\ (v^2 - 1) \tilde{e}_{r_1 \dots r_d} + v \tilde{e}_{r_1 \dots r_{d-1} r_{d+2}}, & \text{if } r_d > n + 1. \end{cases}$$

3. STRUCTURES OF THE SCHUR ALGEBRA \mathbf{S}^j

In this section, we establish some fundamental multiplication formulas for the algebra \mathbf{S}^j and its action on \mathbf{T}_d . Then we establish a monomial basis and a canonical basis for \mathbf{S}^j .

3.1. Relations for \mathbf{S}^j . We shall use the notation $U \stackrel{a}{\subset} W$ to denote that U is a subspace of W of codimension a (for $a = 1, 2$). For $i \in [1, n]$, $a \in [1, n+1]$, and for $V, V' \in X$, we set

$$(3.1) \quad \mathbf{e}_i(V, V') = \begin{cases} v^{-|V'_{i+1}/V'_i|}, & \text{if } V_i \stackrel{1}{\subset} V'_i, V_j = V_{j'}, \forall j \in [1, n] \setminus \{i\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(3.2) \quad \mathbf{f}_i(V, V') = \begin{cases} v^{-|V'_i/V'_{i-1}|}, & \text{if } V_i \stackrel{1}{\supset} V'_i, V_j = V_{j'}, \forall j \in [1, n] \setminus \{i\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(3.3) \quad \mathbf{d}_a^{\pm 1}(V, V') = \begin{cases} v^{\mp(|V'_a/V'_{a-1}|)}, & \text{if } V = V'; \\ 0, & \text{otherwise.} \end{cases}$$

We set $\mathbf{d}_a = \mathbf{d}_a^{+1}$. Clearly, $\mathbf{e}_i, \mathbf{f}_i, \mathbf{d}_a, \mathbf{d}_a^{-1}$ lie in the \mathcal{A} -algebra \mathbf{S}^j .

Let $\bar{}$ be the bar involution on \mathcal{A} and $\mathbb{Q}(v)$ by sending $v \mapsto v^{-1}$. Let

$$[r] = \frac{v^r - v^{-r}}{v - v^{-1}}, \quad \text{for } r \in \mathbb{Z},$$

be the (bar-invariant) quantum integer r . In the following proposition we adopt the convention of dropping the product symbol $*$ to make the formulas more readable.

Proposition 3.1. *The following relations hold in \mathbf{S}^j : for $i, j \in [1, n]$ and $a \in [1, n+1]$,*

$$\begin{aligned} & \mathbf{d}_i \mathbf{d}_i^{-1} = \mathbf{d}_i^{-1} \mathbf{d}_i = 1, \\ & \mathbf{d}_i \mathbf{d}_j = \mathbf{d}_j \mathbf{d}_i, \\ & \mathbf{d}_i \mathbf{e}_j \mathbf{d}_i^{-1} = v^{\delta_{i,j} - \delta_{i,j+1}} \mathbf{e}_j, \\ & \mathbf{d}_i \mathbf{f}_j \mathbf{d}_i^{-1} = v^{-\delta_{i,j} + \delta_{i,j+1}} \mathbf{f}_j, \\ & \mathbf{e}_i \mathbf{f}_j - \mathbf{f}_j \mathbf{e}_i = \delta_{i,j} \frac{\mathbf{d}_i \mathbf{d}_{i+1}^{-1} - \mathbf{d}_i^{-1} \mathbf{d}_{i+1}}{v - v^{-1}}, & \text{if } i, j \neq n, \\ & \mathbf{e}_i^2 \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i^2 = [2] \mathbf{e}_i \mathbf{e}_j \mathbf{e}_i, & \text{if } |i - j| = 1, \\ & \mathbf{f}_i^2 \mathbf{f}_j + \mathbf{f}_j \mathbf{f}_i^2 = [2] \mathbf{f}_i \mathbf{f}_j \mathbf{f}_i, & \text{if } |i - j| = 1, \\ & \mathbf{e}_i \mathbf{e}_j = \mathbf{e}_j \mathbf{e}_i, & \text{if } |i - j| > 1, \\ & \mathbf{f}_i \mathbf{f}_j = \mathbf{f}_j \mathbf{f}_i, & \text{if } |i - j| > 1, \\ (a) \quad & \mathbf{d}_{n+1} \mathbf{d}_i = \mathbf{d}_i \mathbf{d}_{n+1}, \mathbf{d}_{n+1} \mathbf{d}_{n+1}^{-1} = \mathbf{d}_{n+1}^{-1} \mathbf{d}_{n+1} = 1, \\ (a') \quad & \mathbf{d}_{n+1} \mathbf{e}_i \mathbf{d}_{n+1}^{-1} = v^{-2\delta_{n,i}} \mathbf{e}_i, \mathbf{d}_{n+1} \mathbf{f}_i \mathbf{d}_{n+1}^{-1} = v^{2\delta_{n,i}} \mathbf{f}_i, \\ (b) \quad & \mathbf{e}_n^2 \mathbf{f}_n + \mathbf{f}_n \mathbf{e}_n^2 = [2] \left(\mathbf{e}_n \mathbf{f}_n \mathbf{e}_n - \mathbf{e}_n (v \mathbf{d}_n \mathbf{d}_{n+1}^{-1} + v^{-1} \mathbf{d}_n^{-1} \mathbf{d}_{n+1}) \right), \\ (c) \quad & \mathbf{f}_n^2 \mathbf{e}_n + \mathbf{e}_n \mathbf{f}_n^2 = [2] \left(\mathbf{f}_n \mathbf{e}_n \mathbf{f}_n - (v \mathbf{d}_n \mathbf{d}_{n+1}^{-1} + v^{-1} \mathbf{d}_n^{-1} \mathbf{d}_{n+1}) \mathbf{f}_n \right). \end{aligned}$$

Proof. It suffices to prove the formulas when we specialize v to $\mathbf{v} \equiv \sqrt{q}$ and then perform the convolution products over \mathbb{F}_q .

The relations above except the labeled ones are identical to the type A case and hence are verified as in [BLM].

Let us verify (b). Without loss of generality, we assume that $n = 2$. We have

$$\mathbf{e}_2 \mathbf{e}_2 \mathbf{f}_2(V, V') = \begin{cases} \mathbf{v}^{-(2D-3|V_1|-5)}(q+1) \frac{q^{D-2(|V_1|+1)}-1}{q-1}, & \text{if } V_1 \overset{1}{\subset} V'_1, |V'_1| < d; \\ \mathbf{v}^{-(2D-3|V_1|-5)}(q+1), & \text{if } |V_1 \cap V'_1| = |V_1| - 1, |V'_1| \neq d; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{f}_2 \mathbf{e}_2 \mathbf{e}_2(V, V') = \begin{cases} \mathbf{v}^{-(2D-3|V_1|-3)}(q+1) \frac{q^{|V_1|-1}-1}{q-1}, & \text{if } V_1 \overset{1}{\subset} V'_1; \\ \mathbf{v}^{-(2D-3|V_1|-3)}(q+1), & \text{if } |V_1 \cap V'_1| = |V_1| - 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{e}_2 \mathbf{f}_2 \mathbf{e}_2(V, V') = \begin{cases} \mathbf{v}^{-(2D-3|V_1|-4)} \left(\frac{q^{D-2|V_1|-1}-1}{q-1} + \frac{q^{|V_1|+1}-q}{q-1} \right), & \text{if } V_1 \overset{1}{\subset} V'_1; \\ \mathbf{v}^{-(2D-3|V_1|-4)}(q+1), & \text{if } |V_1 \cap V'_1| = |V_1| - 1, |V'_1| \neq d; \\ \mathbf{v}^{-(2D-3|V_1|-4)}, & \text{if } |V_1 \cap V'_1| = |V_1| - 1, |V'_1| = d; \\ 0, & \text{otherwise.} \end{cases}$$

Hence (b) follows.

The involution $(V, V') \mapsto (V', V)$ defines an \mathcal{A} -linear anti-automorphism τ on \mathbf{S}^j such that $\tau(\mathbf{d}_a) = \mathbf{d}_a$ (for $a = 1, 2, \dots, n+1$), and for any $V, V' \in X$,

$$\tau(\mathbf{e}_2)(V, V') = \mathbf{v}^{-(D-3|V_1|+1)} \mathbf{f}_2(V, V'),$$

$$\tau(\mathbf{f}_2)(V, V') = \mathbf{v}^{D-3|V_1|-2} \mathbf{e}_2(V, V').$$

By applying the anti-automorphism τ to (b), we obtain (c). The verifications of (a) and (a') are easy and will be skipped. \square

3.2. Multiplication formulas. For $i, j \in [1, N]$, let E_{ij} be the standard elementary matrix in $\text{Mat}_{N \times N}(\mathbb{N})$. Let

$$(3.4) \quad E_{ij}^\theta = E_{ij} + E_{N+1-i, N+1-j}.$$

The (i, j) -entry of E_{ij}^θ will be denoted by ϵ_{ij}^θ . Note that

$$\epsilon_{ij}^\theta = \begin{cases} 2, & \text{if } i = j = n+1; \\ 1, & \text{otherwise.} \end{cases}$$

Recall that the set $\{e_A \mid A \in \Xi_d\}$ is an \mathcal{A} -basis for \mathbf{S}^j . The following lemma is a counterpart of [BLM, Lemma 3.2].

Lemma 3.2. (a) *For $A, B \in \Xi_d$ such that $\text{ro}(A) = \text{co}(B)$ and $B - E_{h, h+1}^\theta$ is a diagonal matrix for some $h \in [1, n]$, we have*

$$e_B * e_A = \sum_{p \in [1, N], a_{h+1, p} \geq \epsilon_{h+1, p}^\theta} v^{2 \sum_{j>p} a_{hj}} \frac{v^{2(1+a_{hp})} - 1}{v^2 - 1} e_{A+E_{hp}^\theta - E_{h+1, p}^\theta}.$$

- (b) For $A, C \in \Xi_d$ such that $\text{ro}(A) = \text{co}(C)$ and $C - E_{h+1,h}^\theta$ is a diagonal matrix for some $h \in [1, n]$, we have

$$e_C * e_A = \sum_{p \in [1, N], a_{hp} \geq 1} v^{2 \sum_{j < p} a_{h+1,j}} \frac{v^{2(1+a_{h+1,p})} - 1}{v^2 - 1} e_{A - E_{hp}^\theta + E_{h+1,p}^\theta}.$$

Proof. The proof for case (a) with $h \in [1, n]$ and for case (b) with $h \in [1, n-1]$ is essentially the same as the proof of [BLM, Lemma 3.2], and hence will not be repeated here. We shall prove the new case when $h = n$ in (b) as follows. As before, the proof is further reduced to analogous results over finite fields by specializing v to $\mathbf{v} \equiv \sqrt{q}$. Under the assumption of (b) and $h = n$, we have

$$e_C * e_A = \sum_{p \in [1, N], a_{np} \geq 1} \#G_p e_{A - E_{np}^\theta + E_{n+1,p}^\theta},$$

where the set G_p consists of all subspaces S in \mathbb{F}_q^D determined by the following conditions:

- S is isotropic;
- $V_n \subset S$ and $|S/V_n| = 1$;
- $V_n \cap V_j' = S \cap V_j'$ for $j < p$ and $V_n \cap V_j' \neq S \cap V_j'$ for $j \geq p$;
- (V, V') is a fixed pair of flags in X whose associated matrix is $A - E_{np}^\theta + E_{n+1,p}^\theta$.

This is obtained by an argument similar to [BLM, §3.1]. So the problem is reduced to compute the number $\#G_p$.

First, we consider the case when $h = n$ and $p \leq n$. The situation is the same as [BLM] when we observe that the subspace $V_n + V_n^\perp \cap V_j'$ is isotropic if V_n and V_j' are isotropic.

Next, we consider the case when $h = n$ and $p = n+1$. We set

$$G'_{n+1} = \{T \text{ isotropic} \mid V_n + V_n^\perp \cap V_n' \subset T \subseteq V_n + V_n^\perp \cap V_{n+1}'\}.$$

It is clear that

$$\#G'_{n+1} = \frac{q^{a_{n+1,n+1}+2-1} - 1}{q - 1} = \frac{q^{1+a_{n+1,n+1}} - 1}{q - 1}.$$

We define a map $\Psi_{n+1} : G_{n+1} \rightarrow G'_{n+1}$ by $\Psi_{n+1}(S) = S + S^\perp \cap V_n'$. For a fixed $T \in G'_{n+1}$, we can identify W with the vector space $V_n \oplus \frac{V_n + V_n^\perp \cap V_{n+1}'}{V_n} \oplus L$ where L is an isotropic subspace of dimension 1 in $V_n + V_n^\perp \cap V_{n+1}'$. Under such an identification, we see that the fiber $\Psi_{n+1}^{-1}(T)$ is isomorphic to $\frac{V_n + V_n^\perp \cap V_{n+1}'}{V_n}$. So Ψ_{n+1} gives us a vector bundle G_{n+1} over G'_{n+1} with rank equal to $|\frac{V_n + V_n^\perp \cap V_{n+1}'}{V_n}| = \sum_{j < n+1} a_{n+1,j}$. We thus have

$$\#G_{n+1} = \#\Psi_{n+1}^{-1}(T) \cdot \#G'_{n+1} = q^{\sum_{j < n+1} a_{n+1,j}} \frac{q^{1+a_{n+1,n+1}} - 1}{q - 1}.$$

So the formula in (b) holds for $h = n$ and $p = n+1$.

Finally, we consider the case when $h = n$ and $p \geq n+2$. Let G'_p be the set of all flags $W = (W_i)_{1 \leq i \leq n}$ in \mathbb{F}_q^D subject to the following conditions:

- W_i is isotropic for $1 \leq i \leq n$ and $V_n \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_n$.
- $V_n + V_n^\perp \cap V_i' \subset W_i$ and $|\frac{W_i}{V_n + V_n^\perp \cap V_i'}| = 1$ for $1 \leq i \leq N - p$.
- $|V_n + V_n^\perp \cap V_i'| = |W_i|$ and $|\frac{W_i}{V_n + V_n^\perp \cap V_i'}| = 1$ for $N - p + 1 \leq i \leq n$.
- $W_1 \not\subseteq V_n + V_n^\perp \cap V_{p-1}'$.

We define a map $\Psi : G_p \rightarrow G'_p$ by $\Psi(S) = (S + S^\perp \cap V'_i)_{1 \leq i \leq n}$. Let us fix a flag $W = (W_i)_{1 \leq i \leq n}$ in G'_p . Then the subspace W_1 can be rewritten as

$$(3.5) \quad W_1 \simeq V_n \oplus \frac{V_n + V_n^\perp \cap V'_1}{V_n} \oplus \langle w_1 \rangle,$$

where w_1 is a vector not contained in $V_n + V_n^\perp \cap V'_{p-1}$ and $\langle w_1 \rangle$ is the subspace spanned by w_1 . One can check that

$$(3.6) \quad \Psi^{-1}((W_i)_{1 \leq i \leq n}) \simeq \{V_n \oplus \langle w_1 + x \rangle \mid x \in \frac{V_n + V_n^\perp \cap V'_1}{V_n}\} \simeq \mathbb{F}_q^{a_{n+1,1}},$$

if the two vector spaces in (3.5) are identified. This implies that Ψ is surjective and a vector bundle of fiber dimension $a_{n+1,1}$.

Let I_p be the set of all flags $U = (U_{N-p+1} \subseteq \cdots \subseteq U_n)$ subject to the following conditions:

- $V_n + V_n^\perp \cap V'_{N-p} \subset U_{N-p+1} \subseteq V_n + V_n^\perp \cap V'_{N-p+1}$ and $|\frac{V_n + V_n^\perp \cap V'_{N-p+1}}{U_{N-p+1}}| = 1$;
- $V_n + V_n^\perp \cap V'_{i-1} \not\subset U_i \subset V_n + V_n^\perp \cap V_i$ and $|\frac{V_n + V_n^\perp \cap V'_i}{U_i}| = 1$ for $N - p + 2 \leq i \leq n$.

We stratify G'_p as

$$G'_p = \bigsqcup_{U \in I_p} G'_{p,U}, \quad G'_{p,U} = \{W \in G'_p \mid W_i \cap (V_n + V_n^\perp \cap V_j) = U_i, \forall N - p + 1 \leq i \leq n\}.$$

Inside $G'_{p,U}$, the subspace W_n is subject to the conditions:

$$U_n \subset W_n \subset U_n^\perp, \quad W_n \neq V_n + V_n^\perp \cap V'_n \quad \text{and} \quad W_n \not\subset V_n + V_n^\perp \cap V'_{n+1}.$$

The number of choices for such a W_n is

$$\frac{q^{a_{n+1,n+1}+1} - 1}{q - 1} - 1 - q \frac{q^{a_{n+1,n+1}-1} - 1}{q - 1} = q^{a_{n+1,n+1}}.$$

Fixing W_n , we see that the number of choices for W_{n-1} is $q^{|U_n/U_{n-1}|} = q^{a_{n+1,n}}$. Inductively, we have

$$(3.7) \quad \#Z''_{p,U} = \prod_{1 \leq i \leq n} q^{a_{n+1,i+1}}.$$

We now consider the index set I_p . The number of choices for U_{N-p+1} is $\frac{q^{1+a_{n+1,N-p+1}-1}}{q-1}$. Fixing U_{N-p+1} , we see that the number of choices for U_{N-p+2} is $q^{a_{n+1,N-p+2}} = q^{a_{n+1,p}}$. Inductively, we conclude that

$$(3.8) \quad \#I_p = \frac{q^{1+a_{n+1,p}} - 1}{q - 1} \prod_{n+2 \leq i \leq p-1} q^{a_{n+1,i}}.$$

By putting together (3.6), (3.7) and (3.8), we see immediately that

$$\#G_p = q^{a_{n+1,1}} \#G'_p = q^{a_{n+1,1}} \#I_p \#G'_{p,U} = \frac{q^{1+a_{n+1,p}} - 1}{q - 1} \prod_{1 \leq i < p} q^{a_{n+1,i}}.$$

This finishes the proof of the lemma. \square

We set, for $a \in \mathbb{Z}$ and $b \in \mathbb{N}$,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \prod_{1 \leq i \leq b} \frac{v^{2(a-i+1)} - 1}{v^{2i} - 1}, \quad \text{and} \quad [a] = \begin{bmatrix} a \\ 1 \end{bmatrix}.$$

We have the following multiplication formulas for the algebra \mathbf{S}^j , which is an analogue of [BLM, Lemma 3.4(a1),(b1)].

Proposition 3.3. *Suppose that $h \in [1, n]$ and $R \in \mathbb{N}$.*

(a) *For $A, B \in \Xi_d$ such that $B - RE_{h,h+1}^\theta$ is diagonal and $\text{ro}(A) = \text{co}(B)$, we have*

$$(3.9) \quad e_B * e_A = \sum_t v^{2 \sum_{j>u} a_{hj} t_u} \prod_{u=1}^N \begin{bmatrix} a_{hu} + t_u \\ t_u \end{bmatrix} e_{A + \sum_{u=1}^N t_u (E_{hu}^\theta - E_{h+1,u}^\theta)},$$

where $t = (t_1, \dots, t_N) \in \mathbb{N}^N$ with $\sum_{u=1}^N t_u = R$ such that

$$\begin{cases} t_u \leq a_{h+1,u}, & \text{if } h < n, \\ t_u + t_{N+1-u} \leq a_{h+1,u}, & \text{if } h = n. \end{cases}$$

(b) *For $A, C \in \Xi_d$ such that $C - RE_{h+1,h}^\theta$ is diagonal and $\text{co}(C) = \text{ro}(A)$, we have*

$$(3.10) \quad e_C * e_A = \sum_t v^{2 \sum_{j<u} a_{h+1,j} t_u} \prod_{u=1}^N \begin{bmatrix} a_{h+1,u} + t_u \\ t_u \end{bmatrix} e_{A - \sum_{u=1}^N t_u (E_{hu}^\theta - E_{h+1,u}^\theta)}, \text{ for } h < n,$$

and for $h = n$,

$$(3.11) \quad e_C * e_A = \sum_t v^{2 \sum_{j<u} a_{n+1,j} t_u} v^{2 \sum_{N+1-j<u<j} t_u t_j + \sum_{u>n+1} t_u (t_u - 1)} \prod_{u<n+1} \begin{bmatrix} a_{n+1,u} + t_u \\ t_u \end{bmatrix} \\ \cdot \prod_{u>n+1} \begin{bmatrix} a_{n+1,u} + t_u + t_{N+1-u} \\ t_u \end{bmatrix} \prod_{i=0}^{t_{n+1}-1} \frac{[a_{n+1,n+1} + 1 + 2i]}{[i + 1]} e_{A - \sum_{u=1}^N t_u (E_{nu}^\theta - E_{n+1,u}^\theta)},$$

where $t = (t_1, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^N t_u = R$ and $t_u \leq a_{hu}$.

(Note that the above coefficients are in \mathcal{A} since $a_{n+1,n+1}$ is an odd integer.)

Proof. We shall prove (3.11) for $h = n$ by induction on R in detail. It is clear that (3.11) holds for $R = 1$ by Lemma 3.2(b). Let us write C_R instead of C in (b) to indicate the dependence on R . Similarly we let $C_{R+1} \in \Xi_d$ be such that $C_{R+1} - (R+1)E_{h+1,h}^\theta$ is diagonal and $\text{co}(C_{R+1}) = \text{ro}(A)$, and let C_1 be such that $C_1 - E_{h+1,h}^\theta$ is diagonal and $\text{co}(C_1) = \text{ro}(C_R)$. We have by Lemma 3.2(b) again that

$$(3.12) \quad e_{C_1} * e_{C_R} = [R+1] e_{C_{R+1}}.$$

We write $A(t) = A - \sum_{u=1}^N t_u (E_{n,u}^\theta - E_{n+1,u}^\theta)$ and $G_{A,t}$ the coefficient of $A(t)$ in (3.11) for $t = (t_1, \dots, t_N)$. So we have

$$e_{C_1} * e_{C_R} * e_A = \sum_{t,s} G_{A,t} G_{A(t),s} e_{A(t+s)},$$

where the sum runs over all (s, t) such that $\sum t_u = R$ and $\sum s_u = 1$. By a direct computation, for any r such that $\sum r_u = R + 1$, we have

$$\frac{1}{[R+1]} \sum_{t+s=r} G_{A,t} G_{A(t),s} = \frac{1}{[R+1]} \sum_s v^{2\sum_{j<s'} t_j} [t_{s'}] G_{A,r} = G_{A,r},$$

where s' is the unique nonzero position in s . The formula (3.11) follows. The proofs of (3.9) and (3.10) are similar and will be skipped. \square

3.3. The \mathbf{S}^j -action on \mathbf{T}_d . Recall we have an \mathcal{A} -basis $\{e_{r_1 \dots r_d} \mid 1 \leq r_1, \dots, r_d \leq N\}$ for \mathbf{T}_d , and there is a bijection (see (2.6)) between these d -tuples $r_1 \dots r_d$ and $\mathbf{r} = (r_1, \dots, r_D)$ subject to $r_c + r_{D+1-c} = N + 1$. A (simpler) variant of the proof of Lemma 3.2 gives us the following proposition.

Proposition 3.4. *The left \mathbf{S}^j -action on \mathbf{T}_d via the convolution product*

$$\mathbf{S}^j \times \mathbf{T}_d \longrightarrow \mathbf{T}_d$$

is given as follows: for $1 \leq i \leq n$,

$$(3.13) \quad \mathbf{e}_i e_{r_1 \dots r_d} = v^{-\#\{1 \leq k \leq D \mid r_k = i+1\}} \sum_{\{1 \leq p \leq D \mid r_p = i\}} v^{2\#\{1 \leq j < p \mid r_j = i+1\}} e_{r'_1 \dots r'_d},$$

where $\mathbf{r}' = (r'_1, \dots, r'_D)$ for each p with $r_p = i$ satisfies $r'_s = r_s$ (for $s \neq p, D+1-p$), $r'_p = i+1$, and $r'_{D+1-p} = N-i$;

$$(3.14) \quad \mathbf{f}_i e_{r_1 \dots r_d} = v^{-\#\{1 \leq k \leq D \mid r_k = i\}} \sum_{\{1 \leq p \leq D \mid r_p = i+1\}} v^{2\#\{p < j \leq D \mid r_j = i\}} e_{r''_1 \dots r''_d},$$

where $\mathbf{r}'' = (r''_1, \dots, r''_D)$ for each $r_p = i+1$ satisfies $r''_s = r_s$ (for $s \neq p, D+1-p$), $r''_p = i$, and $r''_{D+1-p} = N+1-i$; and $\mathbf{d}_a^\pm e_{r_1 \dots r_d} = v^{\mp \#\{1 \leq j \leq D \mid r_j = a\}} e_{r_1 \dots r_d}$.

3.4. A standard basis. Recall that \mathcal{O}_A is the associated $O(D)$ -orbit of A . We are interested in computing its dimension over the algebraic closure $\overline{\mathbb{F}}_q$. We first recall that the dimension of $O(D)$ is $D(D-1)/2$. Next we shall compute the stabilizer of a point (V, V') in \mathcal{O}_A . We decompose the vector space $\overline{\mathbb{F}}_q^D$ into $\overline{\mathbb{F}}_q^D = \oplus_{1 \leq i, j \leq N} Z_{ij}$ such that

$$V_a = \oplus_{i \leq a, j \in [1, N]} Z_{ij} \quad \text{and} \quad V'_b = \oplus_{i \in [1, N], j \leq b} Z_{ij}, \quad \forall a, b \in [1, N].$$

With respect to the decomposition and the lexicographic order for the set $\{(i, j) \mid i, j \in [1, N]\}$, we can choose the bilinear form Q to be anti-diagonal and identity block matrix on each anti-diagonal position. The Lie algebra of the stabilizer $G_{V, V'}$ of the point (V, V') in $O(D)$ is then the space of all linear maps $x_{(i, j), (k, l)} : Z_{ij} \rightarrow Z_{kl}$ satisfying the following conditions.

- (a) $x_{(i, j), (k, l)} = 0$ unless $i \geq k$ and $j \geq l$.
- (b) $x_{(i, j), (k, l)} = -{}^t x_{(N+1-k, N+1-l), (N+1-i, N+1-j)}$, $\forall i, j, k, l \in [1, N]$.

Note that the condition (a) is obtained as in [BLM, 2.1], while the condition (b) is from the choice of Q . From (b), we see that $x_{(i, j), (k, l)} = -{}^t x_{(i, j), (k, l)}$ if and only if $i+k = N+1$ and $j+l = N+1$. So the dimension of the stabilizer $G_{V, V'}$ is

$$\sum_{\substack{i \geq k, j \geq l \\ i+k < N+1}} a_{ij} a_{kl} + \sum_{\substack{i \geq k, j \geq l \\ i+k = N+1, j+l < N+1}} a_{ij} a_{kl} + \sum_{i \geq n+1, j \geq n+1} a_{ij} (a_{ij} - 1)/2.$$

Summarizing, we have proved the following.

Lemma 3.5. *The dimension of \mathcal{O}_A , denoted by $d(A)$, is given by*

$$d(A) = \sum_{\substack{i < k \text{ or } j < l \\ i+k < N+1}} a_{ij}a_{kl} + \sum_{\substack{i < k \text{ or } j < l \\ i+k=N+1, j+l < N+1}} a_{ij}a_{kl} + \sum_{i < n+1 \text{ or } j < n+1} a_{ij}(a_{ij} - 1)/2.$$

(Here the condition $\substack{i < k \text{ or } j < l \\ i+k < N+1}$ means that $i < k, i+k < N+1$ or $i \geq k, j < l, i+k < N+1$.)

Denote by $r(A)$ the dimension of the image of \mathcal{O}_A under the first projection $X \times X \rightarrow X$. Note that $r(A) = d(B)$, the dimension of the orbit \mathcal{O}_B , where B is a diagonal matrix such that $b_{ii} = \sum_j a_{ij}$. By applying Lemma 3.5 to the matrix B , we have

$$(3.15) \quad r(A) = \sum_{i < k, i+k < N+1} a_{ij}a_{kl} + \sum_{i < n+1} a_{ij}a_{il}/2 - \sum_{i < n+1} a_{ij}/2.$$

By Lemma 3.5 and (3.15), we have

$$(3.16) \quad d(A) - r(A) = \sum_{\substack{i > k, j < l \\ i+k < N+1}} a_{ij}a_{kl} + \sum_{\substack{i < n+1 \text{ or } j < N+1-l \\ j < l}} a_{ij}a_{il} + \sum_{i \geq n+1 > j} a_{ij}(a_{ij} - 1)/2.$$

We set

$$(3.17) \quad [A] = [A]_d = v^{-d(A)+r(A)} e_A, \quad \forall A \in \Xi_d.$$

(The notation $[A]_d$ will only be used when it is necessary to indicate the dependence on d .) Then $\{[A] \mid A \in \Xi_d\}$ forms an \mathcal{A} -basis for \mathbf{S}^j , which we call a *standard basis* of \mathbf{S}^j .

Remark 3.6. It follows by the same argument as for [BLM, Lemma 3.10] that the assignment $[A] \mapsto [{}^t A]$ defines an \mathcal{A} -linear anti-automorphism on \mathbf{S}^j .

The following is a reformulation of the multiplication formulas for \mathbf{S}^j in Proposition 3.3.

Proposition 3.7. (a) *Under the assumptions in Proposition 3.3(a), we have*

$$[B] * [A] = \sum_t v^{\beta(t)} \prod_{u=1}^N \overline{\begin{bmatrix} a_{hu} + t_u \\ t_u \end{bmatrix}} \left[A + \sum_{1 \leq u \leq N} t_u (E_{hu}^\theta - E_{h+1,u}^\theta) \right],$$

where t is summed over as in Proposition 3.3(a) and

$$(3.18) \quad \beta(t) = \sum_{j \leq l} a_{hl} t_j - \sum_{j < l} a_{h+1,l} t_j + \sum_{j < l} t_j t_l + \delta_{h,n} \left(\sum_{\substack{j < l \\ j+l < N+1}} t_j t_l + \sum_{j < n+1} \frac{t_j(t_j + 1)}{2} \right).$$

(b) *Under the assumptions in Proposition 3.3(b), we have*

$$[C] * [A] = \sum_t v^{\beta'(t)} \prod_{u=1}^N \overline{\begin{bmatrix} a_{h+1,u} + t_u \\ t_u \end{bmatrix}} \left[A - \sum_{1 \leq u \leq N} t_u (E_{hu}^\theta - E_{h+1,u}^\theta) \right], \forall h < n,$$

where t is summed over as in Proposition 3.3(b) and

$$(3.19) \quad \beta'(t) = \sum_{j \geq l} a_{h+1,l} t_j - \sum_{j > l} a_{hl} t_j + \sum_{j > l} t_j t_l.$$

For $h = n$, we have

$$(3.20) \quad [C] * [A] = \sum_t v^{\beta''(t)} \prod_{u < n+1} \overline{\begin{bmatrix} a_{n+1,u} + t_u + t_{N+1-u} \\ t_u \end{bmatrix}} \prod_{u > n+1} \overline{\begin{bmatrix} a_{n+1,u} + t_u \\ t_u \end{bmatrix}} \\ \cdot \prod_{i=0}^{t_{n+1}-1} \frac{\overline{[a_{n+1,n+1} + 1 + 2i]}}{[i+1]} \left[A - \sum_{1 \leq u \leq N} t_u (E_{nu}^\theta - E_{n+1,u}^\theta) \right],$$

where

$$(3.21) \quad \beta''(t) = \sum_{j \geq l} a_{h+1,l} t_j - \sum_{j > l} a_{hl} t_j + \sum_{j > l} t_j t_l - \sum_{\substack{j < l \\ j+l < N+1}} t_j t_l - \sum_{j < n+1} \frac{t_j(t_j-1)}{2} + \frac{R(R-1)}{2}.$$

Proof. By Proposition 3.3, we have

$$\beta(t) = d(X) - r(X) - (d(A) - r(A)) - (d(B) - r(B)) + 2 \sum_{j > u} a_{hj} t_u + 2 \sum_u a_{hu} t_u,$$

where

$$X = A + \sum_u t_u (E_{hu}^\theta - E_{h+1,u}^\theta).$$

By direct computations, we have $d(B) - r(B) = \sum_{j,u} a_{hj} t_u$. Then by a lengthy calculation, we have

$$\begin{aligned} & d(X) - r(X) - (d(A) - r(A)) \\ &= \sum_{j < l} a_{hl} t_j - \sum_{j < l} a_{h+1,l} t_j + \sum_{j < l} t_j t_l + \delta_{h,n} \left(\sum_{j < l, j+l < N+1} t_j t_l + \sum_{j < n+1} \frac{t_j(t_j+1)}{2} \right). \end{aligned}$$

So we obtain the formula of $\beta(t)$. The computations for $\beta'(t)$ and $\beta''(t)$ are similar. \square

3.5. A monomial basis. We say that $A \leq B$ if $\mathcal{O}_A \subseteq \overline{\mathcal{O}}_B$ over $\overline{\mathbb{F}}_q$. This defines a partial order \leq in Ξ_d . Following [BLM, 3.5], we define a second partial order \preceq on Ξ_d by declaring $A \preceq B$ if and only if

$$(3.22) \quad \sum_{r \leq i; s \geq j} a_{rs} \leq \sum_{r \leq i; s \geq j} b_{rs}, \quad \forall i < j,$$

$$(3.23) \quad \sum_{r \geq i; s \leq j} a_{rs} \leq \sum_{r \geq i; s \leq j} b_{rs}, \quad \forall i > j.$$

Note that (3.23) is redundant, since it can be deduced from (3.22) and $a_{ij} = a_{N+1-i, N+1-j}$. Since the Bruhat orders on Weyl groups of type A and B are compatible with each other, the next result follows immediately from [BLM, 3.5].

Lemma 3.8. *If $A \leq B$ for $A, B \in \Xi_d$, then we have $A \preceq B$.*

We introduce a partial order \sqsubseteq on Ξ_d as follows: for $A, A' \in \Xi_d$, we say that

$$(3.24) \quad A' \sqsubseteq A \text{ if and only if } A' \preceq A, \text{ ro}(A') = \text{ro}(A) \text{ and } \text{co}(A') = \text{co}(A).$$

We write $A' \sqsubset A$ if $A' \sqsubseteq A$ and $A' \neq A$.

In the expression “ M + lower terms” below, the “lower terms” represents a linear combination of elements strictly less than M with respect to the partial order \sqsubseteq .

Lemma 3.9. *Let R be a positive integer.*

(a) *Suppose that $A \in \Xi_d$ satisfies one of the following conditions:*

$$\begin{aligned} a_{hj} &= 0, \quad \forall j \geq k; \quad a_{h+1,k} = R, a_{h+1,j} = 0, \quad \forall j > k, \quad \text{if } h \in [1, n]; \text{ or} \\ a_{nj} &= 0, \quad \forall j \geq k; \quad a_{n+1,k} = R, a_{n+1,j} = 0, \quad \forall j > k, \quad \text{if } h = n, k \in (n+1, N]; \text{ or} \\ a_{nj} &= 0, \quad \forall j \geq n+1; \quad a_{n+1,n+1} = 2R + a, a_{n+1,j} = 0, \quad \forall j > n+1, \quad \text{if } h = n, k = n+1, \end{aligned}$$

for some odd integer a . Let B be the matrix such that $B - RE_{h,h+1}^\theta$ is diagonal and $\text{co}(B) = \text{ro}(A)$. Then

$$[B] * [A] = [M] + \text{lower terms}, \quad \text{where } M = A + R(E_{h,k}^\theta - E_{h+1,k}^\theta).$$

(b) *Suppose that $A \in \Xi_d$ satisfies one of the following conditions:*

$$\begin{aligned} a_{hj} &= 0, \quad \forall j < k, a_{hk} = R; \quad a_{h+1,j} = 0, \quad \forall j \leq k, \quad \text{if } h \in [1, n]; \text{ or} \\ a_{nj} &= 0, \quad \forall j < k, a_{nk} = R; \quad a_{n+1,j} = 0, \quad \forall j \leq k, \quad \text{if } h = n, k \in [1, n]. \end{aligned}$$

Let $C \in \Xi_d$ be a matrix such that $C - RE_{h+1,h}^\theta$ is diagonal and $\text{co}(C) = \text{ro}(A)$. Then

$$[C] * [A] = [M] + \text{lower terms}, \quad \text{where } M = A - R(E_{h,k}^\theta - E_{h+1,k}^\theta).$$

Proof. Observe that $\beta(t) = \beta'(t) = 0$ in Proposition 3.3 for t such that $t_k = R$ and 0, otherwise. The lemma follows from the same argument as that of [BLM, 3.8] by using again that the partial order \preceq is compatible with the analogous one in [BLM, 3.5]. \square

Theorem 3.10. *For any $A \in \Xi_d$, we have*

$$(3.25) \quad \prod_{1 \leq j \leq h < i \leq N} [D_{i,h,j} + a_{ij}E_{h+1,h}^\theta] = [A] + \text{lower terms}$$

(this element in \mathbf{S}^j will be denoted by m_A),

where the product in $(\mathbf{S}^j, *)$ is taken in the following order: (i, h, j) proceeds (i', h', j') if and only if $i < i'$, or $i = i'$, $j < j'$, or $i = i'$, $j = j'$, $h > h'$; the diagonal matrices $D_{i,h,j} \in \text{Mat}_{N \times N}(\mathbb{N})$ are uniquely determined by $\text{ro}(A)$ and $\text{co}(A)$. Moreover, the product has $\frac{N(N^2-1)}{6}$ terms.

Proof. The proof is a slight modification of the proof of [BLM, Proposition 3.9] by using Lemma 3.9. One just needs to be cautious when $h > n$. In this case, $E_{h+1,h}^\theta = E_{N-h,N+1-h}^\theta$, from which one uses Lemma 3.9(a).

Let us explain the proof in more details for the $n = 2$ (i.e., $N = 5$) case. We start with a diagonal matrix D such that $\text{ro}(D) = \text{co}(D) = \text{co}(A)$. We must fill in each off diagonal entries with the desired number. Since all matrices involved satisfy the property that the entries of (i, j) and $(N+1-i, N+1-j)$ are the same, we only need to fill in all the entries below the diagonal. We do it by multiplying repetitively from the left a certain matrix. Of course, we always need to have a leading term in each step. To make this work, we use Lemma 3.9 and fill the entries below the diagonal from bottom to top and from right to left, which is exactly the order stated in the theorem.

To this end, we shall first fill in the $(5, 4)$ -entry. We multiply $D_{5,4,4} + a_{54}E_{54}^\theta$ by D where a_{ij} is the (i, j) -entry of A and $D_{5,4,4}$ is a diagonal matrix such that $\text{co}(D_{5,4,4} + a_{54}E_{54}^\theta) = \text{ro}(D)$. In particular

$$[D_{5,4,4} + a_{54}E_{54}^\theta] * [D] = [D_{5,4,4} + a_{54}E_{54}^\theta],$$

with the right hand side of the desired number at the $(5, 4)$ -entry. Next, we fill the $(5, 3)$ -entry. We multiply the above product by $[D_{5,3,3} + a_{53}E_{43}^\theta]$ from the left. Since $E_{43}^\theta = E_{23}^\theta$, we obtain by Lemma 3.9(a) that the leading term of the resulting product is of the form:

$$\begin{pmatrix} * & a_{12} & 0 & 0 & 0 \\ 0 & * & a_{13} & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & a_{53} & * & 0 \\ 0 & 0 & 0 & a_{54} & * \end{pmatrix}$$

To bring down a_{53} to the desired position, we multiply the above matrix by $[D_{5,4,3} + a_{53}E_{54}^\theta]$ from the left. By Lemma 3.9(b), the leading term for the resulting product is

$$\begin{pmatrix} * & a_{12} & a_{13} & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & a_{53} & a_{54} & * \end{pmatrix}.$$

Summarizing, in order to put a_{53} in the $(5, 3)$ -entry, we need the piece

$$[D_{5,4,3} + a_{53}E_{54}^\theta] * [D_{5,3,3} + a_{53}E_{43}^\theta].$$

We can now apply repetitively the above procedure to the rest of the entries in the prescribed order. In particular, for a_{52} , we need the piece

$$[D_{5,4,2} + a_{52}E_{54}^\theta] * [D_{5,3,2} + a_{52}E_{43}^\theta] * [D_{5,2,2} + a_{52}E_{32}^\theta].$$

For a_{51} , we need the piece

$$[D_{5,4,1} + a_{51}E_{54}^\theta] * [D_{5,3,1} + a_{51}E_{43}^\theta] * [D_{5,2,1} + a_{51}E_{32}^\theta] * [D_{5,1,1} + a_{51}E_{21}^\theta].$$

For a_{43} , we need the piece

$$[D_{4,3,3} + a_{43}E_{43}^\theta].$$

For a_{42} , we need the piece

$$[D_{4,3,2} + a_{42}E_{43}^\theta] * [D_{4,2,2} + a_{42}E_{32}^\theta].$$

For a_{41} , we need the piece

$$[D_{4,3,1} + a_{41}E_{43}^\theta] * [D_{4,2,1} + a_{41}E_{32}^\theta] * [D_{4,1,1} + a_{41}E_{21}^\theta].$$

For a_{32} , we need

$$[D_{3,2,2} + a_{32}E_{32}^\theta].$$

For a_{31} , we need

$$[D_{3,2,1} + a_{31}E_{32}^\theta] * [D_{3,1,1} + a_{31}E_{21}^\theta].$$

For a_{21} , we need

$$[D_{2,1,1} + a_{21}E_{21}^\theta].$$

By putting the pieces together, we have the theorem for $n = 2$ and the general case follows in the same pattern.

It follows by Remark 3.11 below that the number of the terms in the product (3.25) is half of what is in [BLM, 3.9(a)]. \square

Remark 3.11. The ordering of the product (3.25) coincides with the one used in [DDPW, Theorem 13.24]. Namely, if the superscript θ in (3.25) is dropped, the left hand side becomes exactly the second half of a similar product in [DDPW, Theorem 13.24] (which is basically [BLM, 3.9(a)]). As explained in [DDPW, Notes for §13.7, pp.589] and the reference therein, the ordering of the products adopted in [DDPW, Theorem 13.24] is not the same as the one used in [BLM, 3.9(a)], but the resulting products are the same.

Then by Theorem 3.10 the transition matrix from $\{m_A \mid A \in \Xi_d\}$ to the standard basis $\{[A] \mid A \in \Xi_d\}$ is unital triangular, and hence $\{m_A \mid A \in \Xi_d\}$ forms an \mathcal{A} -basis of \mathbf{S}^j , which we call a *monomial basis* of \mathbf{S}^j .

The following lemma follows by definitions.

Lemma 3.12. *For $i \in [1, n], a \in [1, n+1]$, we have the following identities in \mathbf{S}^j :*

$$\mathbf{f}_i = \sum_B [B], \quad \mathbf{e}_i = \sum_C [C], \quad \mathbf{d}_a = \sum_D v^{-D_{aa}} [D]$$

where the sums are over $B, C, D \in \Xi_d$ such that $B - E_{i,i+1}^\theta$, $C - E_{i+1,i}^\theta$, and D are diagonal, respectively.

The following corollary of Theorem 3.10 is now immediate by applying Lemma 3.12 and a standard Vandermonde-determinant-type argument.

Corollary 3.13. *The \mathbf{e}_i , \mathbf{f}_i , $\mathbf{d}_i^{\pm 1}$ and $\mathbf{d}_{i+1}^{\pm 1}$ for $i \in [1, n]$ generate the $\mathbb{Q}(v)$ -algebra $\mathbb{Q}\mathbf{S}^j$.*

Bearing in mind the presentations of the standard v -Schur algebras of type A (cf. [DDPW]), we expect a presentation of the $\mathbb{Q}(v)$ -algebra $\mathbb{Q}\mathbf{S}^j$ with generators given in the above corollary subject to relations in Proposition 3.1 together with the following additional relations:

$$\begin{aligned} \mathbf{d}_{n+1} \mathbf{d}_n^2 \cdots \mathbf{d}_1^2 &= v^{-D}, \\ (\mathbf{d}_i - 1)(\mathbf{d}_i - v^{-1})(\mathbf{d}_i - v^{-2}) \cdots (\mathbf{d}_i - v^{-d}) &= 0, \quad \forall i \in [1, n], \\ (\mathbf{d}_{n+1} - v^{-1}) \cdots (\mathbf{d}_{n+1} - v^{-D}) &= 0. \end{aligned}$$

3.6. A canonical basis. Let IC_A , for $A \in \Xi_d$, be the shifted intersection complex associated with the closure of the orbit \mathcal{O}_A such that the restriction of IC_A to \mathcal{O}_A is the constant sheaf on \mathcal{O}_A . Since IC_A is $O(D)$ -equivariant, the stalks of the i -th cohomology sheaf of IC_A at different points in $\mathcal{O}_{A'}$ (for $A' \in \Xi_d$) are isomorphic. Let $\mathcal{H}_{\mathcal{O}_{A'}}^i(\mathrm{IC}_A)$ denote the stalk of the i -th cohomology group of IC_A at any point in $\mathcal{O}_{A'}$. We set

$$\begin{aligned} P_{A',A} &= \sum_{i \in \mathbb{Z}} \dim \mathcal{H}_{\mathcal{O}_{A'}}^i(\mathrm{IC}_A) v^{i-d(A)+d(A')}, \\ (3.26) \quad \{A\} &= \sum_{A' \leq A} P_{A',A} [A']. \end{aligned}$$

When it is necessary to indicate the dependence on d of $\{A\}$, we will sometimes write $\{A\}_d$ for $\{A\}$. By the properties of intersection complexes, we have

$$(3.27) \quad P_{A,A} = 1, \quad P_{A',A} \in v^{-1}\mathbb{N}[v^{-1}] \text{ if } A' < A.$$

As in [BLM, 1.4], we have an anti-linear bar involution $\bar{} : \mathbf{S}^j \rightarrow \mathbf{S}^j$ such that

$$\bar{v} = v^{-1}, \quad \overline{\{A\}} = \{A\}, \quad \forall A \in \Xi_d.$$

In particular, we have

$$\overline{[A]} = \sum_{A' \leq A} c_{A',A} [A'], \quad \text{where } c_{A,A} = 1, c_{A',A} \in \mathbb{Z}[v, v^{-1}].$$

Then $\mathbf{B}_d^j := \{\{A\} \mid A \in \Xi_d\}$ forms an \mathcal{A} -basis for \mathbf{S}^j , called a *canonical basis*.

The approach to the canonical basis for \mathbf{S}^j above follows [BLM], and it can also be done following an alternative algebraic approach developed by Du (see [Du92]).

3.7. An inner product. We set

$$d_A = d(A) - r(A).$$

Then, recalling ${}^t A$ denotes the transpose of A we have

$$(3.28) \quad 2(d_A - d_{{}^t A}) = \frac{1}{2} \left(\sum_{i=1}^N \text{ro}(A)_i^2 - \text{co}(A)_i^2 \right) - \frac{1}{2} (\text{ro}(A)_{n+1} - \text{co}(A)_{n+1}).$$

Given $A, A' \in \Xi_d$, fix any element L' in $X_{\text{co}(A)}$ and set $X_{t_A}^{L'} = \{L \mid (L', L) \in \mathcal{O}_{t_A}\}$. A standard argument shows that $\#X_{t_A}^{L'}$ is the specialization at $v = \sqrt{q}$ of a Laurent polynomial $f_{A,A'}(v) \in \mathbb{Z}[v, v^{-1}]$. Following McGerty [Mc12], we define a bilinear form

$$(\cdot, \cdot)_D : \mathbf{S}^j \times \mathbf{S}^j \longrightarrow \mathbb{Q}(v)$$

by

$$(e_A, e_{A'})_D = \delta_{A,A'} v^{2(d_A - d_{{}^t A})} f_{A,A'}(v).$$

By using (3.28) and arguing in exactly the same manner as that of [Mc12, Proposition 3.2], we have the following.

Proposition 3.14. $([A]e_{A_1}, e_{A_2})_D = v^{d_A - d_{{}^t A}} (e_{A_1}, [{}^t A]e_{A_2})_D$, for all $A, A_1, A_2 \in \Xi_d$.

We record the following useful consequence of Proposition 3.14.

Corollary 3.15. Let $i \in [1, n]$, $a \in [1, n+1]$, and let $A_1, A_2 \in \Xi_d$. Then we have

- (a) $(\mathbf{e}_i e_{A_1}, e_{A_2})_D = (e_{A_1}, v^{-1} \mathbf{d}_i^{-1} \mathbf{d}_{i+1} \mathbf{f}_i e_{A_2})_D$;
- (b) $(\mathbf{f}_i e_{A_1}, e_{A_2})_D = (e_{A_1}, v \mathbf{e}_i \mathbf{d}_i \mathbf{d}_{i+1}^{-1} e_{A_2})_D$;
- (c) $(\mathbf{d}_a e_{A_1}, e_{A_2})_D = (e_{A_1}, \mathbf{d}_a e_{A_2})_D$.

The same argument as for [Mc12, Lemma 3.5] now gives us the following.

Proposition 3.16. (a) $([A], [A])_D \in 1 + v^{-1} \mathbb{Z}[v^{-1}]$, for any $A \in \Xi_d$.
 (b) $([A], [A'])_D = 0$ if $A \neq A'$.

The next proposition follows from Proposition 3.16 together with the definition and property of the canonical basis given in (3.26) and (3.27).

Proposition 3.17. (a) The canonical basis \mathbf{B}_d^j satisfies the almost orthonormality, i.e., $(\{A\}, \{A'\})_D \in \delta_{A,A'} + v^{-1} \mathbb{Z}[v^{-1}]$, for any $A, A' \in \Xi_d$.
 (b) The signed canonical basis $(-\mathbf{B}_d^j) \cup \mathbf{B}_d^j$ of the \mathcal{A} -module \mathbf{S}^j is characterized by the almost orthonormal property (a) together with the bar-invariance.

4. THE ALGEBRA \mathbf{K}^J AND ITS IDENTIFICATION AS A COIDEAL ALGEBRA

In this section we construct an \mathcal{A} -algebra \mathbf{K}^J out of \mathbf{S}^J from a stabilization procedure. We then show that \mathbf{K}^J is isomorphic to an integral form of a modified coideal algebra $\dot{\mathbf{U}}^J$. The canonical bases for \mathbf{K}^J and $\dot{\mathbf{U}}^J$ are constructed. A geometric realization of the $(\mathbf{U}^J, \mathbf{H}_{B_d})$ -duality is established.

4.1. Stabilization. Below we shall imitate [BLM, §4] to develop a stabilization procedure to construct a limit \mathcal{A} -algebra \mathbf{K}^J out of \mathbf{S}^J as d goes to ∞ . As the constructions are largely the same as *loc. cit.*, we will be sketchy.

Recall $N = 2n + 1$. We introduce the set

$$(4.1) \quad \tilde{\Xi} = \{A = (a_{ij}) \in \text{Mat}_{N \times N}(\mathbb{Z}) \mid a_{ij} \geq 0 \ (i \neq j), \\ a_{n+1, n+1} \in 2\mathbb{Z} + 1, a_{ij} = a_{N+1-i, N+1-j} \ (\forall i, j)\}.$$

Let \mathbf{K}^J be the free \mathcal{A} -module with an \mathcal{A} -basis given by the symbols $[A]$, for $A \in \tilde{\Xi}$. Also set

$$(4.2) \quad \tilde{\Xi}^{\text{diag}} = \{A \in \tilde{\Xi} \mid A \text{ is diagonal}\}, \quad \Xi := \cup_{d \geq 0} \Xi_d.$$

For $A \in \tilde{\Xi}$, setting ${}_{2p}A := A + 2pI$ we have ${}_{2p}A \in \Xi$ for integers $p \gg 0$. Given matrices $A_1, A_2, \dots, A_f \in \tilde{\Xi}$ (with the same total sum of entries), one shows exactly as in [BLM, 4.2] that there exists $Z_i \in \tilde{\Xi}$ ($1 \leq i \leq m$) such that

$$(4.3) \quad [{}_{2p}A_1] * [{}_{2p}A_2] * \dots * [{}_{2p}A_f] = \sum_{i=1}^m G_i(v, v^{-2p}) [{}_{2p}Z_i],$$

where $G_i(v, v')$ lies in a subring \mathcal{R}_1 of $\mathbb{Q}(v)[v']$ as defined in [BLM, 4.1]. This allows us to define a unique structure of associative \mathcal{A} -algebra (without unit) on \mathbf{K}^J (with product denoted by \cdot) such that

$$(4.4) \quad [A_1] \cdot [A_2] \cdot \dots \cdot [A_f] = \sum_{i=1}^m G_i(v, 1) [Z_i].$$

From the above stabilization procedure, the multiplication formula in Proposition 3.7(a) leads to the following. For any $A, B \in \tilde{\Xi}$ such that $\text{ro}(A) = \text{co}(B)$ and $B - RE_{h, h+1}^\theta$ is diagonal for some $h \in [1, n]$, we have

$$(4.5) \quad [B] \cdot [A] = \sum_t v^{\beta(t)} \prod_{u=1}^N \overline{\begin{bmatrix} a_{hu} + t_u \\ t_u \end{bmatrix}} \left[A + \sum_u t_u (E_{hu}^\theta - E_{h+1, u}^\theta) \right],$$

where $\beta(t)$ is defined in (3.18) and $t = (t_1, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^N t_u = R$ and $t_u \leq a_{h+1, u}$ for $u \neq h+1$ and $h < n$ or $t_u + t_{N+1-u} \leq a_{n+1, u}$ for $u \neq n+1$ and $h = n$.

Similarly, we obtain the following multiplication formula from the one in Proposition 3.7(b) via the above stabilization procedure. For any $A, C \in \tilde{\Xi}$ such that $\text{ro}(A) = \text{co}(C)$ and $C - RE_{h+1, h}^\theta$ is diagonal for some $h \in [1, n-1]$, we have

$$(4.6) \quad [C] \cdot [A] = \sum_t v^{\beta'(t)} \prod_{u=1}^N \overline{\begin{bmatrix} a_{h+1, u} + t_u \\ t_u \end{bmatrix}} \left[A - \sum_u t_u (E_{hu}^\theta - E_{h+1, u}^\theta) \right],$$

where $\beta'(t)$ is defined in (3.19) and $t = (t_1, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^N t_u = R$ and $0 \leq t_u \leq a_{h,u}$ for $u \neq h$. For $h = n$, we have

$$(4.7) \quad [C] \cdot [A] = \sum_t v^{\beta''(t)} \prod_{u < n+1} \overline{\begin{bmatrix} a_{n+1,u} + t_u + t_{N+1-u} \\ t_u \end{bmatrix}} \prod_{u > n+1} \overline{\begin{bmatrix} a_{n+1,u} + t_u \\ t_u \end{bmatrix}} \\ \cdot \prod_{i=0}^{t_{n+1}-1} \frac{\overline{[a_{n+1,n+1} + 1 + 2i]}}{[i+1]} \left[A - \sum_{u=1}^N t_u (E_{nu}^\theta - E_{n+1,u}^\theta) \right],$$

where $\beta''(t)$ is defined in (3.21) and $t = (t_1, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^N t_u = R$ and $0 \leq t_u \leq a_{n,u}$ for $u \neq n$.

We extend the partial order \sqsubseteq on Ξ_d to $\tilde{\Xi}$ by the same definition (3.24) with $A, A' \in \tilde{\Xi}$. Now it follows by Theorem 3.10 that

$$(4.8) \quad \prod_{1 \leq j \leq h < i \leq N} [D_{i,h,j} + a_{ij} E_{h+1,h}^\theta] = [A] + \sum_{A' \sqsubset A} \gamma_{A',A} [A'], \quad \text{for } \gamma_{A',A} \in \mathcal{A}$$

(this element in \mathbf{K}^j will be denoted by \mathbf{M}_A),

where the product is taken in the same order as specified in Theorem 3.10 and the notation $D_{i,h,j}$ can be found therein. Then $\{\mathbf{M}_A \mid A \in \tilde{\Xi}\}$ forms an \mathcal{A} -basis (called the *monomial basis*) of \mathbf{K}^j . Summarizing, we have established the following.

Proposition 4.1. *The formula (4.4) endows \mathbf{K}^j a structure of associative \mathcal{A} -algebra (without unit). Moreover, this \mathcal{A} -algebra structure on \mathbf{K}^j is characterized by the multiplication formulas (4.5)–(4.7).*

4.2. A canonical basis of \mathbf{K}^j . Just as in [BLM, 4.3, 4.5(b)], we have an anti-linear bar involution on \mathbf{K}^j induced from the ones on \mathbf{S}^j (as d goes to ∞). More explicitly, one shows the following stabilization phenomenon of the bar involutions on \mathbf{S}^j :

$$\overline{[{}_{2p}A]} = \sum_{i=1}^m H_i(v, v^{-2p}) [{}_{2p}T_i] \quad \forall p \gg 0.$$

where $H_i(v, v') \in \mathbb{Q}(v)[v', v'^{-1}]$. Following *loc. cit.*, we obtain an anti-linear bar involution $\bar{\cdot} : \mathbf{K}^j \rightarrow \mathbf{K}^j$ defined by

$$\overline{[A]} = \sum_{i=1}^m H_i(v, 1) [T_i].$$

In particular, we have

$$\overline{[A]} = [A] + \sum_{A' : A' \sqsubset A, A' \neq A} \tau_{A',A} [A'], \quad \text{for } \tau_{A',A} \in \mathcal{A}.$$

By a standard argument (see, e.g., [Lu93, 24.2.1]), we have the following.

Proposition 4.2. *There exists a unique \mathcal{A} -basis $\mathbf{B}^j = \{\{A\} \mid A \in \tilde{\Xi}\}$ for \mathbf{K}^j such that*

$$\overline{\{A\}} = \{A\}, \\ \{A\} = [A] + \sum_{A' \sqsubset A} \pi_{A',A} [A'], \quad \text{for } \pi_{A',A} \in v^{-1}\mathbb{Z}[v^{-1}].$$

The basis \mathbf{B}^j is called the canonical basis of \mathbf{K}^j .

4.3. Definition of \mathbf{U}^j . The algebra \mathbf{U}^j is defined to be the associative algebra over $\mathbb{Q}(v)$ generated by e_i, f_i, d_a, d_a^{-1} , $i = 1, 2, \dots, n$, $a = 1, 2, \dots, n+1$ subject to the following relations, for $i, j = 1, 2, \dots, n$, $a, b = 1, 2, \dots, n+1$:

$$(4.9) \quad \left\{ \begin{array}{ll} d_a d_a^{-1} = d_a^{-1} d_a = 1, \\ d_a d_b = d_b d_a, \\ d_a e_j d_a^{-1} = v^{-\delta_{a,j+1} - \delta_{N+1-a,j+1} + \delta_{a,j}} e_j, \\ d_a f_j d_a^{-1} = v^{-\delta_{a,j} + \delta_{a,j+1} + \delta_{N+1-a,j+1}} f_j, \\ e_i f_j - f_j e_i = \delta_{i,j} \frac{d_i d_{i+1}^{-1} - d_i^{-1} d_{i+1}}{v - v^{-1}}, & \text{if } i, j \neq n, \\ e_i^2 e_j + e_j e_i^2 = \llbracket 2 \rrbracket e_i e_j e_i, & \text{if } |i - j| = 1, \\ f_i^2 f_j + f_j f_i^2 = \llbracket 2 \rrbracket f_i f_j f_i, & \text{if } |i - j| = 1, \\ e_i e_j = e_j e_i, & \text{if } |i - j| > 1, \\ f_i f_j = f_j f_i, & \text{if } |i - j| > 1, \\ f_n^2 e_n + e_n f_n^2 = \llbracket 2 \rrbracket \left(f_n e_n f_n - (v d_n d_{n+1}^{-1} + v^{-1} d_n^{-1} d_{n+1}) f_n \right), \\ e_n^2 f_n + f_n e_n^2 = \llbracket 2 \rrbracket \left(e_n f_n e_n - e_n (v d_n d_{n+1}^{-1} + v^{-1} d_n^{-1} d_{n+1}) \right). \end{array} \right.$$

We also write $e_i = f_{N+1-i}$ and $f_i = e_{N+1-i}$ for $n+1 < i \leq N$. Denote by ${}^0\mathbf{U}^j$ the $\mathbb{Q}(v)$ -subalgebra of \mathbf{U}^j generated by $d_a^{\pm 1}$ for all $a = 1, 2, \dots, n+1$.

Remark 4.3. The $\mathbb{Q}(v)$ -subalgebra of \mathbf{U}^j generated by $e_i, f_i, d_i^{\pm 1}, d_{i+1}^{\pm 1}$, $i = 1, 2, \dots, n-1$ is naturally isomorphic to the quantum group $\mathbf{U}(\mathfrak{gl}(n))$. The algebra \mathbf{U}^j and the quantum group $\mathbf{U}(\mathfrak{gl}(N))$ form the quantum symmetric pair $(\mathbf{U}(\mathfrak{gl}(N)), \mathbf{U}^j)$. This is a variant of the quantum symmetric pair associated with the quantum group $\mathbf{U}(\mathfrak{sl}(N))$; see [Le02, K14] and [BW13, Section 6]. The algebra \mathbf{U}^j (and \mathbf{U}^i in later sections) also appeared independently in [ES13]. The relation between the algebra \mathbf{U}^j defined here and the algebra of the same notation defined in [BW13, Section 6] is just the usual \mathfrak{sl} vs \mathfrak{gl} relation and can be described as follows: the generators e_i and f_i match the generators with $e_{\alpha_{n-i+\frac{1}{2}}}$ and $f_{\alpha_{n-i+\frac{1}{2}}}$ respectively, while

$$d_a d_{a+1}^{-1} = \begin{cases} k_{\alpha_{n-a+\frac{1}{2}}}, & \text{if } 1 \leq a < n; \\ v k_{\alpha_{\frac{1}{2}}}, & \text{if } a = n. \end{cases}$$

The convention on the generators d_a (and in particular on d_{n+1}) is made to better match the geometric construction.

Lemma 4.4. (cf. [BW13, Lemma 6.1]) *The algebra \mathbf{U}^j has an anti-linear bar involution, denoted by $\bar{}$, such that $\overline{d_a} = d_a^{-1}$, $\overline{e_i} = e_i$, and $\overline{f_i} = f_i$ for $i = 1, \dots, n$ and $a = 1, \dots, n+1$.*

Denote by $\mathbf{U}(\mathfrak{gl}(N))$ the quantum general linear Lie algebra of rank N , which is $\mathbb{Q}(v)$ -algebra generated by $E_i, F_i, K_i^{\pm 1}$, and $K_{i+1}^{\pm 1}$, for $i \in [1, N-1]$ subject to a standard set of relations which can be found as part of the relations (4.9) in different notations. (Recall from Remark 4.3 that the quantum group $\mathbf{U}(\mathfrak{gl}(n))$ is naturally a subalgebra of \mathbf{U}^j with the corresponding subset of relations from (4.9).)

Proposition 4.5. *There is an injective $\mathbb{Q}(v)$ -algebra homomorphism $j : \mathbf{U}^j \rightarrow \mathbf{U}(\mathfrak{gl}(N))$ given by, for all $i = 1, \dots, n$,*

$$\begin{aligned} d_i &\mapsto K_i^{-1} K_{N+1-i}^{-1}, & e_i &\mapsto F_i + K_i^{-1} K_{i+1} E_{N-i}, \\ d_{n+1} &\mapsto v K_{n+1}^{-2}, & f_i &\mapsto E_i K_{N-i}^{-1} K_{N+1-i} + F_{N-i}. \end{aligned}$$

Proof. This is a $\mathfrak{gl}(N)$ -variant of [BW13, Proposition 6.2]; also see [K14, ES13]. \square

Following [BLM] and [Lu93, §23.1], we shall similarly define the modified quantum algebra $\dot{\mathbf{U}}^j$ from \mathbf{U}^j , where the unit of \mathbf{U}^j is replaced by a collection of orthogonal idempotents. We shall denote by $\lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ a matrix in $\tilde{\Xi}^{\text{diag}}$. For $\lambda, \lambda' \in \tilde{\Xi}^{\text{diag}}$, we set

$${}_{\lambda} \mathbf{U}_{\lambda'}^j = \mathbf{U}^j / \left(\sum_{a=1}^{n+1} (d_a - v^{-\lambda_a}) \mathbf{U}^j + \sum_{a=1}^{n+1} \mathbf{U}^j (d_a - v^{-\lambda'_a}) \right).$$

Let $\pi_{\lambda, \lambda'} : \mathbf{U}^j \rightarrow {}_{\lambda} \mathbf{U}_{\lambda'}^j$ be the canonical projection. Set

$$\dot{\mathbf{U}}^j = \bigoplus_{\lambda, \lambda' \in \tilde{\Xi}^{\text{diag}}} {}_{\lambda} \mathbf{U}_{\lambda'}^j.$$

Let $D_{\lambda} := \pi_{\lambda, \lambda}(1)$. Following [Lu93, 23.1], $\dot{\mathbf{U}}^j$ is naturally an associative $\mathbb{Q}(v)$ -algebra containing D_{λ} as orthogonal idempotents, and $\dot{\mathbf{U}}^j$ is naturally a \mathbf{U}^j -bimodule. In particular, we have

$$\dot{\mathbf{U}}^j = \sum_{\lambda \in \tilde{\Xi}^{\text{diag}}} \mathbf{U}^j D_{\lambda} = \sum_{\lambda \in \tilde{\Xi}^{\text{diag}}} D_{\lambda} \mathbf{U}^j.$$

By construction, the following relations hold:

$$\begin{aligned} D_{\lambda} D_{\lambda'} &= \delta_{\lambda, \lambda'} D_{\lambda}, \\ d_a D_{\lambda} &= D_{\lambda} d_a = v^{-\lambda_a} D_{\lambda}. \end{aligned}$$

For any $I = (i_1, i_2, \dots, i_j)$ with $i_k \in \{1, 2, \dots, n, n+2, \dots, N\}$, we set $\mathcal{E}_I = e_{i_1} e_{i_2} \dots e_{i_j}$, where $e_{\emptyset} = 1$. Following [K14, Proposition 6.2], there exists a collection of such indices I , denoted by \mathbb{I} , such that $\{\mathcal{E}_I \mid I \in \mathbb{I}\}$ forms a basis of the free ${}^0 \mathbf{U}^j$ -module \mathbf{U}^j . Therefore the elements $\mathcal{E}_I D_{\lambda}$ ($I \in \mathbb{I}, \lambda \in \tilde{\Xi}^{\text{diag}}$) form a basis of the $\mathbb{Q}(v)$ -vector space $\dot{\mathbf{U}}^j$.

4.4. A presentation of $\dot{\mathbf{U}}^j$. Given $\lambda \in \tilde{\Xi}^{\text{diag}}$, we introduce the short-hand notation $\lambda - \alpha_i = \lambda + E_{i+1, i+1}^{\theta} - E_{i, i}^{\theta}$ and $\lambda + \alpha_i = \lambda - E_{i+1, i+1}^{\theta} + E_{i, i}^{\theta}$, for $1 \leq i \leq n$. We define \mathbf{A} to be the $\mathbb{Q}(v)$ -algebra generated by the symbols D_{λ} , $e_i D_{\lambda}$, $D_{\lambda} e_i$, $f_i D_{\lambda}$ and $D_{\lambda} f_i$, for $i = 1, \dots, n$ and

$\lambda \in \tilde{\Xi}^{\text{diag}}$, subject to the following relations (4.10), for $i, j = 1, \dots, n$ and $\lambda, \lambda' \in \tilde{\Xi}^{\text{diag}}$:

$$(4.10) \quad \left\{ \begin{array}{ll} xD_{\lambda}D_{\lambda'}x' = \delta_{\lambda,\lambda'}xD_{\lambda}x', & \text{for } x, x' \in \{1, e_i, e_j, f_i, f_j\}, \\ e_iD_{\lambda} = D_{\lambda-\alpha_i}e_i, \\ f_iD_{\lambda} = D_{\lambda+\alpha_i}f_i, \\ e_iD_{\lambda}f_j = f_jD_{\lambda-\alpha_i-\alpha_j}e_i, & \text{if } i \neq j, \\ e_iD_{\lambda}f_i = f_iD_{\lambda-2\alpha_i}e_i + [\lambda_{i+1} - \lambda_i]D_{\lambda-\alpha_i}, & \text{if } i \neq n, \\ (e_i^2e_j + e_j e_i^2)D_{\lambda} = [2]e_i e_j e_i D_{\lambda}, & \text{if } |i-j| = 1, \\ (f_i^2 f_j + f_j f_i^2)D_{\lambda} = [2]f_i f_j f_i D_{\lambda}, & \text{if } |i-j| = 1, \\ e_i e_j D_{\lambda} = e_j e_i D_{\lambda}, & \text{if } |i-j| > 1, \\ f_i f_j D_{\lambda} = f_j f_i D_{\lambda}, & \text{if } |i-j| > 1, \\ (f_n^2 e_n - [2]f_n e_n f_n + e_n f_n^2)D_{\lambda} = -[2]\left(v^{\lambda_{n+1}-\lambda_n-2} + v^{\lambda_n-\lambda_{n+1}+2}\right)f_n D_{\lambda}, \\ (e_n^2 f_n - [2]e_n f_n e_n + f_n e_n^2)D_{\lambda} = -[2]\left(v^{\lambda_{n+1}-\lambda_n+1} + v^{\lambda_n-\lambda_{n+1}-1}\right)e_n D_{\lambda}. \end{array} \right.$$

To simplify the notation, we shall write $x_1 D_{\lambda^1} x_2 D_{\lambda^2} \cdots x_l D_{\lambda^l} = x_1 x_2 \cdots x_l D_{\lambda^l}$, if the product is not zero; in this case such $\lambda^1, \lambda^2, \dots, \lambda^{l-1}$ are unique.

Proposition 4.6. *We have an isomorphism of $\mathbb{Q}(v)$ -algebras $\dot{\mathbf{U}}^j \cong \mathbf{A}$ by identifying the generators in the same notation. That is, the relations (4.10) are the defining relations of the $\mathbb{Q}(v)$ -algebra $\dot{\mathbf{U}}^j$.*

Proof. By definitions of $\dot{\mathbf{U}}^j$ and \mathbf{U}^j with relations (4.9) in §4.3, we see that $\dot{\mathbf{U}}^j$ satisfies the same relations (4.10) as for \mathbf{A} . Hence there is a surjective algebra homomorphism $\mathbf{A} \rightarrow \dot{\mathbf{U}}^j$, sending generators to generators in the same notation. Following the presentation of the algebra \mathbf{A} in (4.10), we see that the algebra \mathbf{A} has a natural \mathbf{U}^j -bimodule structure, where the actions of $e_i, f_i \in \mathbf{U}^j$ are defined in the obvious way, and the action of d_a on the idempotents D_{λ} is given by $D_{\lambda}d_a = d_a D_{\lambda} = v^{-\lambda_a} D_{\lambda}$. As a left or right \mathbf{U}^j -module, \mathbf{A} is generated by the idempotents D_{λ} , for $\lambda \in \tilde{\Xi}^{\text{diag}}$. Hence we have $\mathbf{A} = \sum_{\lambda \in \tilde{\Xi}^{\text{diag}}} \mathbf{U}^j D_{\lambda}$.

Recall that $\{\mathcal{E}_I \mid I \in \mathbb{I}\}$ forms a basis of the free ${}^0\mathbf{U}^j$ -module \mathbf{U}^j . Therefore we have $\mathbf{A} = \sum_{I \in \mathbb{I}, \lambda \in \tilde{\Xi}^{\text{diag}}} \mathbb{Q}(v)\mathcal{E}_I D_{\lambda}$. Since $\{\mathcal{E}_I D_{\lambda} \mid I \in \mathbb{I}, \lambda \in \tilde{\Xi}^{\text{diag}}\}$ forms a $\mathbb{Q}(v)$ -basis of $\dot{\mathbf{U}}^j$, these elements in the same notation in \mathbf{A} must be linearly independent, and hence the homomorphism $\mathbf{A} \rightarrow \dot{\mathbf{U}}^j$ is an isomorphism. \square

4.5. Isomorphism ${}_{\mathcal{A}}\dot{\mathbf{U}}^j \cong \mathbf{K}^j$. The following theorem provides a geometric realization of $\dot{\mathbf{U}}^j$ thanks to the geometric nature of \mathbf{K}^j .

Theorem 4.7. *We have an isomorphism of $\mathbb{Q}(v)$ -algebras $\aleph : \dot{\mathbf{U}}^j \rightarrow {}_{\mathbb{Q}}\mathbf{K}^j$ which sends*

$$e_i D_{\lambda} \mapsto [D_{\lambda} - E_{i,i}^{\theta} + E_{i+1,i}^{\theta}], \quad f_i D_{\lambda} \mapsto [D_{\lambda} - E_{i+1,i+1}^{\theta} + E_{i,i+1}^{\theta}],$$

and $D_{\lambda} \mapsto [D_{\lambda}]$, for all $i = 1, \dots, n$, and $\lambda \in \tilde{\Xi}^{\text{diag}}$.

Proof. Via a direct computation we can check using the multiplication formulas (4.5)–(4.7) that the relations (4.10) for $\dot{\mathbf{U}}^j$ are satisfied by the images of $D_{\lambda}, e_i D_{\lambda}, f_i D_{\lambda}$ as specified in the lemma. Since the relations (4.10) are defining relations for $\dot{\mathbf{U}}^j$ by Proposition 4.6, we conclude that \aleph is an algebra homomorphism.

It remains to show \aleph is a linear isomorphism. Set $\varepsilon_{n+1} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^{2n+1}$, where 1 is in the $(n+1)$ st position. Note that ε_{n+1} , when regarded as in $\tilde{\Xi}^{\text{diag}}$, is simply the

diagonal matrix $E_{n+1,n+1}$. Also set

$$\begin{aligned}\tilde{\Theta} &= \{A = (a_{ij}) \in \text{Mat}_{N \times N}(\mathbb{Z}) \mid a_{ij} \geq 0 \ (i \neq j)\}, \\ \tilde{\Theta}^- &= \{A \in \tilde{\Theta} \mid a_{ij} = 0 \ (i < j), \text{co}(A) = 0\}, \\ \tilde{\Xi}^- &= \{A \in \tilde{\Xi} \mid \text{co}(A) = \varepsilon_{n+1}\}.\end{aligned}$$

The diagonal entries of a matrix $A' \in \tilde{\Theta}^-$ (respectively, $A \in \tilde{\Xi}^-$) are completely determined by its strictly lower triangular entries. Hence, there is a natural bijection $\tilde{\Theta}^- \longleftrightarrow \tilde{\Xi}^-$, which sends $A' \in \tilde{\Theta}^-$ to $A \in \tilde{\Xi}^-$ such that the strictly lower triangular parts of A and A' are identical.

Denote by \mathbf{K} the \mathcal{A} -algebra in the type A stabilization of [BLM], and ${}_{\mathbb{Q}}\mathbf{K} = \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{K}$. The quantum group $\mathbf{U} = \mathbf{U}(\mathfrak{gl}(N))$ has a triangular decomposition $\mathbf{U} = \mathbf{U}^- \mathbf{U}^0 \mathbf{U}^+$. Denote by $\dot{\mathbf{U}} = \dot{\mathbf{U}}(\mathfrak{gl}(N))$ the modified quantum group of $\mathfrak{gl}(N)$ with idempotents $\mathbf{1}_{\lambda}$ and denote its \mathcal{A} -form by ${}_{\mathcal{A}}\dot{\mathbf{U}}$. It was shown in [BLM] that there exists a $\mathbb{Q}(v)$ -algebra isomorphism $\aleph^a : \dot{\mathbf{U}} \rightarrow {}_{\mathbb{Q}}\mathbf{K}$ (entirely parallel to the homomorphism $\aleph : \dot{\mathbf{U}}^j \rightarrow {}_{\mathbb{Q}}\mathbf{K}^j$ above). Here and below we will add superscript a to indicate type A and distinguish from the notations already used in the type B setting of this paper. Recall \mathbf{K} has a monomial \mathcal{A} -basis $\{{}^a\mathbf{M}_{A'} \mid A' \in \tilde{\Theta}\}$ which is given in [BLM, 4.6(c)] (without such notation or terminology), entirely parallel to the monomial basis for \mathbf{K}^j which we defined in §4.1. Via the isomorphism \aleph^a , the monomial \mathcal{A} -basis $\{{}^a\mathbf{M}_{A'} \mid A' \in \tilde{\Theta}^-\}$ for the \mathcal{A} -submodule $\aleph^a({}_{\mathcal{A}}\mathbf{U}^-\mathbf{1}_0)$ of \mathbf{K} corresponds to a monomial \mathcal{A} -basis $\{{}^u\mathbf{M}_{A'} \mid A' \in \tilde{\Theta}^-\}$ for the \mathcal{A} -submodule ${}_{\mathcal{A}}\mathbf{U}^-\mathbf{1}_0$ of ${}_{\mathcal{A}}\dot{\mathbf{U}}$, where $\aleph^a({}^u\mathbf{M}_{A'}) = {}^a\mathbf{M}_{A'}$.

Note that we may regard that $e_i, f_i \in \mathbf{U}^j$ have “leading terms” F_i, F_{N-i} by Proposition 4.5. By [Le02] (see [K14, Proposition 6.2]), replacing the leading terms F_i, F_{N-i} by e_i and f_i respectively, we obtain a monomial $\mathbb{Q}(v)$ -basis $\{\tilde{\mathbf{M}}_A \mid A \in \tilde{\Xi}^-\}$ for $\dot{\mathbf{U}}^j D_{\varepsilon_{n+1}}$ from the monomial basis $\{{}^u\mathbf{M}_{A'} \mid A' \in \tilde{\Theta}^-\}$ for $\mathbf{U}^-\mathbf{1}_0$. The homomorphism $\aleph : \dot{\mathbf{U}}^j \rightarrow {}_{\mathbb{Q}}\mathbf{K}^j$ restricts to a $\mathbb{Q}(v)$ -linear map $\aleph|_{\varepsilon_{n+1}} : \dot{\mathbf{U}}^j D_{\varepsilon_{n+1}} \rightarrow {}_{\mathbb{Q}}\mathbf{K}^j[D_{\varepsilon_{n+1}}]$, which sends $\tilde{\mathbf{M}}_A$ to \mathbf{M}_A for $A \in \tilde{\Xi}^-$ (using Remark 3.11 and the definition (4.8) of a monomial basis element as a product of generators). Hence $\aleph|_{\varepsilon_{n+1}}$ is a $\mathbb{Q}(v)$ -linear isomorphism. This leads to a $\mathbb{Q}(v)$ -linear isomorphism $\aleph|_{\lambda} : \dot{\mathbf{U}}^j D_{\lambda} \rightarrow {}_{\mathbb{Q}}\mathbf{K}^j[D_{\lambda}]$ (which is a restriction of \aleph), for any $\lambda \in \tilde{\Xi}^{\text{diag}}$, via the following commutative diagram:

$$\begin{array}{ccc}\dot{\mathbf{U}}^j D_{\varepsilon_{n+1}} & \xrightarrow{\aleph|_{\varepsilon_{n+1}}} & {}_{\mathbb{Q}}\mathbf{K}^j[D_{\varepsilon_{n+1}}] \\ \#_{\lambda} \downarrow & & \#_{\lambda} \downarrow \\ \dot{\mathbf{U}}^j D_{\lambda} & \xrightarrow{\aleph|_{\lambda}} & {}_{\mathbb{Q}}\mathbf{K}^j[D_{\lambda}]\end{array}$$

Here $\#_{\lambda} : \mathbf{K}^j[D_{\varepsilon_{n+1}}] \rightarrow \mathbf{K}^j[D_{\lambda}]$ is a $\mathbb{Q}(v)$ -linear isomorphism which sends a monomial basis element $\mathbf{M}_{A+\varepsilon_{n+1}}$ to $\mathbf{M}_{A+\lambda}$, and $\#_{\lambda} : \dot{\mathbf{U}}^j D_{\varepsilon_{n+1}} \rightarrow \dot{\mathbf{U}}^j D_{\lambda}$ is defined accordingly.

Putting $\aleph|_{\lambda}$ together, we have shown that $\aleph : \dot{\mathbf{U}}^j \rightarrow {}_{\mathbb{Q}}\mathbf{K}^j$ is an isomorphism. \square

The bar involution on \mathbf{U}^j (given in Lemma 4.4) induces a compatible bar involution on $\dot{\mathbf{U}}^j$, denoted also by $\bar{}$, which fixes all the generators $D_{\lambda}, e_i D_{\lambda}, f_i D_{\lambda}$.

Corollary 4.8. *The homomorphism \aleph intertwines the bar involutions on $\dot{\mathbf{U}}^j$ and on ${}_{\mathbb{Q}}\mathbf{K}^j$, i.e., $\aleph(\bar{u}) = \overline{\aleph(u)}$, for $u \in \mathbf{U}^j$.*

Proof. The corollary follows by checking when u is a generator of $\dot{\mathbf{U}}^j$. \square

We define an \mathcal{A} -subalgebra of $\dot{\mathbf{U}}^j$ by ${}_{\mathcal{A}}\dot{\mathbf{U}}^j := \aleph^{-1}(\mathbf{K}^j)$. Clearly we have ${}_{\mathcal{A}}\dot{\mathbf{U}}^j \otimes_{\mathcal{A}} \mathbb{Q}(v) = \dot{\mathbf{U}}^j$.

Corollary 4.9. *The integral form ${}_{\mathcal{A}}\dot{\mathbf{U}}^j$ is a free \mathcal{A} -submodule of $\dot{\mathbf{U}}^j$ and it is stable under the bar involution.*

The isomorphism $\aleph : {}_{\mathcal{A}}\dot{\mathbf{U}}^j \rightarrow \mathbf{K}^j$ allows us to transport the canonical basis \mathbf{B}^j for \mathbf{K}^j to a canonical basis for ${}_{\mathcal{A}}\dot{\mathbf{U}}^j$ (and for $\dot{\mathbf{U}}^j$). Introduce the divided powers $e_i^{(r)} = e_i^r / [r]!$ and $f_i^{(r)} = f_i^r / [r]!$, for $r \geq 1$, where $[r]! = [r] \cdots [1]$.

Proposition 4.10. *The isomorphism $\aleph : \dot{\mathbf{U}}^j \rightarrow {}_{\mathbb{Q}}\mathbf{K}^j$ satisfies, for $r \geq 1$,*

$$\aleph(e_i^{(r)} D_{\lambda}) = [D_{\lambda} - rE_{i,i}^{\theta} + rE_{i+1,i}^{\theta}] \quad \text{and} \quad \aleph(f_i^{(r)} D_{\lambda}) = [D_{\lambda} - rE_{i+1,i+1}^{\theta} + rE_{i,i+1}^{\theta}].$$

Moreover, the \mathcal{A} -algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}^j$ is generated by $e_i^{(r)} D_{\lambda}$ and $f_i^{(r)} D_{\lambda}$ for various i, r and λ .

Proof. We shall only show the first identity, as the second one is entirely similar. Following the multiplication formula (4.5), we have

$$[D_{\lambda'} - E_{i,i}^{\theta} + E_{i+1,i}^{\theta}][D_{\lambda''} - (r-1)E_{i,i}^{\theta} + (r-1)E_{i+1,i}^{\theta}] = [r][D_{\lambda} - rE_{i,i}^{\theta} + rE_{i+1,i}^{\theta}],$$

where $\text{ro}(D_{\lambda'} - E_{i,i}^{\theta} + E_{i+1,i}^{\theta}) = \text{ro}(D_{\lambda} - rE_{i,i}^{\theta} + rE_{i+1,i}^{\theta})$ and $\text{co}(D_{\lambda''} - (r-1)E_{i,i}^{\theta} + (r-1)E_{i+1,i}^{\theta}) = \text{co}(D_{\lambda} - rE_{i,i}^{\theta} + rE_{i+1,i}^{\theta})$. The first statement follows.

The second statement follows from Theorem 3.10 and the definition of ${}_{\mathcal{A}}\dot{\mathbf{U}}^j$. \square

4.6. Homomorphism from \mathbf{K}^j to \mathbf{S}^j . Now we establish a precise relation between the algebras \mathbf{K}^j and \mathbf{S}^j .

Proposition 4.11. *There exists a unique surjective \mathcal{A} -algebra homomorphism $\phi_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$ such that, for $R \geq 0$, $i \in [1, n]$ and $D \in \tilde{\Xi}^{\text{diag}}$,*

$$\begin{aligned} \phi_d([D + RE_{i,i+1}^{\theta}]) &= \begin{cases} [D + RE_{i,i+1}^{\theta}], & \text{if } D + RE_{i,i+1}^{\theta} \in \Xi_d; \\ 0, & \text{otherwise;} \end{cases} \\ \phi_d([D + RE_{i+1,i}^{\theta}]) &= \begin{cases} [D + RE_{i+1,i}^{\theta}], & \text{if } D + RE_{i+1,i}^{\theta} \in \Xi_d; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence we have a surjective \mathcal{A} -algebra homomorphism $\phi_d \circ \aleph : \dot{\mathbf{U}}^j \rightarrow \mathbf{S}^j$.

Proof. The existence of a homomorphism of $\mathbb{Q}(v)$ -algebras $\phi_d : {}_{\mathbb{Q}}\mathbf{K}^j \rightarrow {}_{\mathbb{Q}}\mathbf{S}^j$ (or equivalently, a homomorphism $\phi_d \circ \aleph : {}_{\mathbb{Q}}\dot{\mathbf{U}}^j \rightarrow {}_{\mathbb{Q}}\mathbf{S}^j$) given by the above formulas with $R = 0, 1$ follows immediately from Proposition 3.1 and the presentation of $\dot{\mathbf{U}}^j$ (and of ${}_{\mathbb{Q}}\mathbf{K}^j$) from Proposition 4.6 and Theorem 4.7. The surjectivity and uniqueness of such \aleph are clear.

It follows by the multiplication formulas for \mathbf{K}^j and for \mathbf{S}^j that ϕ_d matches the “divides powers”, as indicated by the formulas in the proposition. Now the proposition follows by noting that these “divided powers” (which corresponds to the divided powers in ${}_{\mathcal{A}}\dot{\mathbf{U}}^j$ by Proposition 4.10) generate \mathbf{K}^j and \mathbf{S}^j . \square

4.7. A geometric duality. Recall the embedding $j : \mathbf{U}^j \rightarrow \mathbf{U}(\mathfrak{gl}(N))$ in Proposition 4.5. Let \mathbb{V} be the natural representation of $\mathbf{U}(\mathfrak{gl}(N))$ with the standard basis $\{\mathbf{v}_1, \dots, \mathbf{v}_N\}$. Then $\mathbb{V}^{\otimes d}$ becomes a $\mathbf{U}(\mathfrak{gl}(N))$ -module via the coproduct:

$$\begin{aligned} \Delta : \mathbf{U}(\mathfrak{gl}(N)) &\longrightarrow \mathbf{U}(\mathfrak{gl}(N)) \otimes \mathbf{U}(\mathfrak{gl}(N)), \\ E_i &\mapsto 1 \otimes E_i + E_i \otimes K_i K_{i+1}^{-1}, \\ F_i &\mapsto F_i \otimes 1 + K_i^{-1} K_{i+1} \otimes F_i, \\ K_a &\mapsto K_a \otimes K_a, \end{aligned}$$

for $i = 1, \dots, n$ and $a = 1, \dots, n+1$. Then \mathbb{V} and $\mathbb{V}^{\otimes d}$ are naturally \mathbf{U}^j -modules via the embedding $j : \mathbf{U}^j \rightarrow \mathbf{U}(\mathfrak{gl}(N))$. Following [Lu93, 23.1], $\mathbb{V}^{\otimes d}$ becomes a $\dot{\mathbf{U}}^j$ -module as well.

We write $\mathbf{v}_{r_1 \dots r_d} = \mathbf{v}_{r_1} \otimes \dots \otimes \mathbf{v}_{r_d}$. There is a right action of the Iwahori-Hecke algebra \mathbf{H}_{B_d} on the $\mathbb{Q}(v)$ -vector space $\mathbb{V}^{\otimes d}$ as follows:

$$(4.11) \quad \mathbf{v}_{r_1 \dots r_d} T_j = \begin{cases} v \mathbf{v}_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j < r_{j+1}; \\ v^2 \mathbf{v}_{r_1 \dots r_d}, & \text{if } r_j = r_{j+1}; \\ (v^2 - 1) \mathbf{v}_{r_1 \dots r_d} + v \mathbf{v}_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j > r_{j+1}, \end{cases}$$

for $1 \leq j \leq d-1$, and

$$(4.12) \quad \mathbf{v}_{r_1 \dots r_{d-1} r_d} T_d = \begin{cases} v \mathbf{v}_{r_1 \dots r_{d-1} r_{d+2}}, & \text{if } r_d < n+1; \\ v^2 \mathbf{v}_{r_1 \dots r_{d-1} r_d}, & \text{if } r_d = n+1; \\ (v^2 - 1) \mathbf{v}_{r_1 \dots r_{d-1} r_d} + v \mathbf{v}_{r_1 \dots r_{d-1} r_{d+2}}, & \text{if } r_d > n+1, \end{cases}$$

where $r_{d+2} = N+1 - r_{d-1}$. The $(\dot{\mathbf{U}}^j, \mathbf{H}_{B_d})$ -duality established in [BW13, Theorem 6.26], states that we have commuting actions of $\dot{\mathbf{U}}^j$ and ${}_{\mathbb{Q}}\mathbf{H}_{B_d}$ on $\mathbb{V}^{\otimes d}$ which form double centralizers. We caution the reader that the conventions of the algebras \mathbf{U}^j and \mathbf{H}_{B_d} formulated above are chosen to fit with the geometric counterpart (see Proposition 4.12 below) and they differ from those in *loc. cit.*

Recall the right action of \mathbf{H}_{B_d} on \mathbf{T}_d from Lemma 2.4. The left action of \mathbf{S}^j on \mathbf{T}_d given in §2.3 is lifted to a left action of \mathbf{K}^j on \mathbf{T}_d via the homomorphism $\phi_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$. We define a $\mathbb{Q}(v)$ -vector space isomorphism:

$$(4.13) \quad \begin{aligned} \Omega : \mathbb{V}^{\otimes d} &\longrightarrow {}_{\mathbb{Q}}\mathbf{T}_d \\ \mathbf{v}_{r_1} \otimes \mathbf{v}_{r_2} \otimes \dots \otimes \mathbf{v}_{r_d} &\mapsto \tilde{e}_{r_1 r_2 \dots r_d}. \end{aligned}$$

The following provides a geometric realization of the $(\dot{\mathbf{U}}^j, \mathbf{H}_{B_d})$ -duality established in [BW13, Theorem 6.26].

Proposition 4.12. *We have the following commutative diagram of double centralizing actions under the identification $\Omega : \mathbb{V}^{\otimes d} \longrightarrow {}_{\mathbb{Q}}\mathbf{T}_d$:*

$$\begin{array}{ccccc} {}_{\mathbb{Q}}\mathbf{K}^j & \curvearrowright & {}_{\mathbb{Q}}\mathbf{T}_d & \curvearrowright & {}_{\mathbb{Q}}\mathbf{H}_{B_d} \\ \mathfrak{N} \uparrow & & \Omega \uparrow & & \parallel \\ \dot{\mathbf{U}}^j & \curvearrowright & \mathbb{V}^{\otimes d} & \curvearrowright & {}_{\mathbb{Q}}\mathbf{H}_{B_d} \end{array}$$

A general consideration (cf. [P09]) also shows that the actions of ${}_{\mathbb{Q}}\mathbf{S}^j$ and ${}_{\mathbb{Q}}\mathbf{H}_{B_d}$ on ${}_{\mathbb{Q}}\mathbf{T}_d$ satisfy the double centralizer property (under the assumption that $n \geq d$).

Remark 4.13. The \imath canonical basis of the \mathbf{U}^J -module $\mathbb{V}^{\otimes d}$ was constructed in [BW13], and it does not coincide with Lusztig's canonical basis of $\mathbb{V}^{\otimes d}$ (regarded as a $\mathbf{U}(\mathfrak{gl}(N))$ -module). An \imath canonical basis on \mathbf{T}_d can also be defined via a standard intersection complex construction (similar to the one on \mathbf{S}^J). These two \imath canonical bases coincide under the isomorphism Ω .

Remark 4.14. The (geometric) symmetric Howe duality in Remark 2.3 can also afford an algebraic formulation. Note that an (algebraic) skew Howe duality was formulated in [ES13] though the double centralizer property was not proved therein (actually this property can be proved easily using \imath Schur duality and a deformation argument).

5. A SECOND GEOMETRIC CONSTRUCTION IN TYPE B

In this section, we provide a realization of a second Schur algebra \mathbf{S}^i (which is a subalgebra of \mathbf{S}^J) in the geometric framework of type B . We establish a generating set and prove a number of relations of \mathbf{S}^i . The algebra \mathbf{S}^i contains a subtle new generator \mathbf{t} , which makes \mathbf{S}^i quite different from \mathbf{S}^J . We show that the monomial and canonical bases of \mathbf{S}^J restrict to monomial and canonical bases of its subalgebra \mathbf{S}^i .

5.1. The setup. Recall Ξ_d, Π from §2.2 and $\tilde{\Xi}$ from (4.1). Let ${}^i\Xi_d$ (and respectively, ${}^i\tilde{\Xi}$) be a subset of Ξ_d (and respectively, $\tilde{\Xi}$) consisting of matrices A in Ξ_d (and respectively, in $\tilde{\Xi}$) such that all the entries in the $(n+1)$ st row and in the $(n+1)$ st column are 0 except $a_{n+1, n+1} = 1$. Also set

$${}^i\tilde{\Xi}^{\text{diag}} = \{A \in {}^i\tilde{\Xi} \mid A \text{ is diagonal}\}, \quad {}^i\Xi_d^{\text{diag}} = \{A \in {}^i\Xi_d \mid A \text{ is diagonal}\}.$$

Similarly, we denote by ${}^i\tilde{\Pi}$ the subset of Π consisting of matrices $B = (b_{ij}) \in \Pi$ such that all the entries in the $(n+1)$ st row and in the $(d+1)$ st column are 0 except $b_{n+1, d+1} = 1$. This extra condition makes a matrix in ${}^i\Xi_d, {}^i\tilde{\Xi}$ or ${}^i\Pi$ look like

$$\begin{pmatrix} * & \cdots & * & 0 & * & \cdots & * \\ * & \cdots & * & 0 & * & \cdots & * \\ \cdots & & & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ * & \cdots & * & 0 & * & \cdots & * \\ * & \cdots & * & 0 & * & \cdots & * \end{pmatrix}.$$

Recall X from §2.2. We consider the following subset of X :

$${}^iX = \{V = (V_i) \in X \mid V_n \text{ is maximal isotropic}\}.$$

The products ${}^iX \times {}^iX$ and ${}^iX \times Y$ are invariant under the diagonal action of $O(D)$. Note that V_n being maximal isotropic for $V = (V_i) \in {}^iX$ (and $V_{n+1} = V_n^\perp$) implies that V_{n+1}/V_n is one-dimensional. Therefore Lemma 2.1 readily leads to the following.

Lemma 5.1. *The bijections in Lemma 2.1 induce the following bijections:*

$$O(D) \backslash {}^iX \times {}^iX \longleftrightarrow {}^i\Xi_d, \quad \text{and} \quad O(D) \backslash {}^iX \times Y \longleftrightarrow {}^i\Pi.$$

The following is a counterpart of Lemma 2.2 and we skip the similar argument.

Lemma 5.2. *We have*

$$\# \, {}^i\Xi_d = \binom{2n^2 + d - 1}{d}, \quad \text{and} \quad \# \, {}^i\Pi = (2n)^d.$$

We define

$$\mathbf{S}^i = \mathcal{A}_{O(D)}({}^iX \times {}^iX), \quad \mathbf{T}_d^i = \mathcal{A}_{O(D)}({}^iX \times Y)$$

to be the space of $O(D)$ -invariant \mathcal{A} -valued functions on ${}^iX \times {}^iX$ and ${}^iX \times Y$, respectively. Following §2.3, under the convolution product \mathbf{S}^i is an \mathcal{A} -subalgebra of \mathbf{S}^j , \mathbf{T}_d^i is a left \mathbf{S}^i -submodule and also a right \mathbf{H}_{B_d} -submodule of \mathbf{T}_d , and we obtain commuting actions of \mathbf{S}^i and \mathbf{H}_{B_d} on \mathbf{T}_d^i . In particular, \mathbf{S}^i is a free \mathcal{A} -module with a basis $\{e_A \mid A \in {}^i\Xi_d\}$ and with a standard basis $\{[A] \mid A \in {}^i\Xi_d\}$ (inherited from their counterparts in \mathbf{S}^j by restriction). Denote

$$\mathbb{Q}\mathbf{S}^i = \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{S}^i, \quad \mathbb{Q}\mathbf{T}_d^i = \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{T}_d^i.$$

5.2. Relations for \mathbf{S}^i . We can still define the elements $\mathbf{e}_i, \mathbf{f}_i$ and \mathbf{d}_a in \mathbf{S}^i for $i \in [1, n-1]$ and $a \in [1, n]$ as done for \mathbf{S}^j in §3.1. However, the elements \mathbf{e}_n and \mathbf{f}_n defined for \mathbf{S}^j will no longer make sense here. Instead, we define a new element $\mathbf{t} \in \mathbf{S}^i$ by setting, for $V, V' \in {}^iX$,

$$(5.1) \quad \mathbf{t}(V, V') = \begin{cases} v^{-(|V'_n| - |V'_{n-1}|)}, & \text{if } |V_n \cap V'_n| \geq d-1, V_j = V'_j, \forall j \in [1, n-1]; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 5.3. One checks that, for $V, V' \in {}^iX$, (and recall that $|V'_{n+1}/V'_n| = 1$),

$$(5.2) \quad \mathbf{t}(V, V') = \mathbf{f}_n \mathbf{e}_n(V, V') - \delta_{V, V'} [|V'_n/V'_{n-1}| - |V'_{n+1}/V'_n|].$$

Proposition 5.4. *The elements $\mathbf{e}_i, \mathbf{f}_i, \mathbf{d}_i^{\pm 1}, \mathbf{d}_{i+1}^{\pm 1}$ for $i \in [1, n-1]$, and \mathbf{t} in \mathbf{S}^i satisfy the following relations:*

- (a) *the defining relations of $\mathbf{U}(\mathfrak{gl}(n))$ for $\mathbf{e}_i, \mathbf{f}_i, \mathbf{d}_i^{\pm 1}, \mathbf{d}_{i+1}^{\pm 1}, \forall i$ (see Remark 4.3);*
- (b) $\mathbf{d}_i \mathbf{t} = \mathbf{t} \mathbf{d}_i, \quad \forall i \in [1, n];$
- (c) $\mathbf{e}_i \mathbf{t} = \mathbf{t} \mathbf{e}_i, \quad \mathbf{f}_i \mathbf{t} = \mathbf{t} \mathbf{f}_i, \quad \forall i \in [1, n-2];$
- (d) $\mathbf{e}_{n-1}^2 \mathbf{t} - [2] \mathbf{e}_{n-1} \mathbf{t} \mathbf{e}_{n-1} + \mathbf{t} \mathbf{e}_{n-1}^2 = 0;$
- (e) $\mathbf{f}_{n-1}^2 \mathbf{t} - [2] \mathbf{f}_{n-1} \mathbf{t} \mathbf{f}_{n-1} + \mathbf{t} \mathbf{f}_{n-1}^2 = 0;$
- (f) $\mathbf{t}^2 \mathbf{e}_{n-1} - [2] \mathbf{t} \mathbf{e}_{n-1} \mathbf{t} + \mathbf{e}_{n-1} \mathbf{t}^2 = \mathbf{e}_{n-1};$
- (g) $\mathbf{t}^2 \mathbf{f}_{n-1} - [2] \mathbf{t} \mathbf{f}_{n-1} \mathbf{t} + \mathbf{f}_{n-1} \mathbf{t}^2 = \mathbf{f}_{n-1}.$

Proof. It suffices to prove the formulas when we specialize v to $\mathbf{v} \equiv \sqrt{q}$ and then perform the convolution products over \mathbb{F}_q .

The relations (a), (b) and (c) are clear. The remaining relations can be reduced to Proposition 3.1 by using (5.2). Below let us give a direct proof.

We now prove (d). Without loss of generality, we may and shall assume that $n = 2$. A direct computation yields

$$\mathbf{e}_1^2 \mathbf{t}(V, V') = \begin{cases} \mathbf{v}^{-3(d-|V_1|)+6} (\mathbf{v} + \mathbf{v}^{-1}), & \text{if } V_1 \subset V'_1 \subset V_2, |V_2 \cap V'_2| \geq d-1, \\ 0, & \text{otherwise;} \end{cases}$$

$$\begin{aligned} \mathbf{te}_1^2(V, V') &= \begin{cases} \mathbf{v}^{-3(d-|V_1|)+4}(\mathbf{v} + \mathbf{v}^{-1}), & \text{if } V_1 \stackrel{2}{\subset} V'_1, |V_2 \cap V'_2| \geq d-1, \\ 0, & \text{otherwise;} \end{cases} \\ \mathbf{e}_1 \mathbf{te}_1(V, V') &= \begin{cases} \mathbf{v}^{-3(d-|V_1|)+5}(\mathbf{v} + \mathbf{v}^{-1}), & \text{if } V_1 \stackrel{2}{\subset} V'_1 \subseteq V_2 \cap V'_2, |V_2 \cap V'_2| \geq d-1, \\ \mathbf{v}^{-3(d-|V_1|)+4}, & \text{if } V_1 \stackrel{2}{\subset} V'_1 \not\subseteq V_2 \cap V'_2, |V_2 \cap V'_2| \geq d-1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The relation (d) follows.

Similarly, the relation (f) is obtained by the following computations:

$$\begin{aligned} \mathbf{t}^2 \mathbf{e}_1(V, V') &= \begin{cases} \mathbf{v}^{-3(d-|V_1|)+1} \frac{q^{d-|V_1|+1}-1}{q-1}, & \text{if } V_1 \stackrel{1}{\subset} V'_1, V_2 = V'_2, \\ \mathbf{v}^{-3(d-|V_1|)+1}(q+1), & \text{if } V_1 \stackrel{1}{\subset} V'_1, |V_2 \cap V'_2| = d-1 \text{ or } d-2, \\ 0, & \text{otherwise;} \end{cases} \\ \mathbf{e}_1 \mathbf{t}^2(V, V') &= \begin{cases} \mathbf{v}^{-3(d-|V_1|)+3} \frac{q^{d-|V_1|}-1}{q-1}, & \text{if } V_1 \stackrel{1}{\subset} V'_1, V_2 = V'_2, \\ \mathbf{v}^{-3(d-|V_1|)+3}(q+1), & \text{if } V_1 \stackrel{1}{\subset} V'_1 \subset V_2, |V_2 \cap V'_2| = d-1 \text{ or } d-2, \\ 0, & \text{otherwise.} \end{cases} \\ \mathbf{te}_1 \mathbf{t}(V, V') &= \begin{cases} \mathbf{v}^{-3(d-|V_1|)+2} \frac{q^{d-|V_1|}-1}{q-1}, & \text{if } V_1 \stackrel{1}{\subset} V'_1, V_2 = V'_2, \\ \mathbf{v}^{-3(d-|V_1|)+2}(q+1), & \text{if } V_1 \stackrel{1}{\subset} V'_1 \subset V_2, |V_2 \cap V'_2| = d-1 \text{ or } d-2, \\ \mathbf{v}^{-3(d-|V_1|)+2}, & \text{if } V_1 \stackrel{1}{\subset} V'_1 \not\subset V_2, |V_2 \cap V'_2| = d-1 \text{ or } d-2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, the involution $(V, V') \mapsto (V', V)$ on ${}^iX \times {}^iX$ induces an anti-involution σ on \mathbf{S}^i such that $\sigma(\mathbf{e}_1) = \mathbf{v}^{d+1}\mathbf{f}_1$, $\sigma(\mathbf{d}_1) = \mathbf{d}_1$, $\sigma(\mathbf{d}_2) = \mathbf{d}_2$, and $\sigma(\mathbf{t}) = \mathbf{t}$. This implies that the relations (e) and (g) follow from (d) and (f), respectively. \square

5.3. A sheaf-theoretic description of \mathbf{t} . We shall now give a geometric interpretation of the element \mathbf{t} . To do this, we need to divert from the previous setting over a finite field to a setting over the complex field \mathbb{C} . Let ${}^iX(\mathbb{C})$ and $Y(\mathbb{C})$ be the algebraic varieties defined over \mathbb{C} , analogous to iX and Y , respectively. Let $S(\tau)$ be the closed subvariety in ${}^iX(\mathbb{C}) \times {}^iX(\mathbb{C})$ defined by

$$S(\tau) = \{(V, V') \in {}^iX(\mathbb{C}) \times {}^iX(\mathbb{C}) \mid |V_n \cap V'_n| \geq d-1, V_j = V'_j, \forall j \in [1, n-1]\}.$$

It is clear that $S(\tau)$ is the subvariety corresponding to the support of \mathbf{t} .

Lemma 5.5. *The variety $S(\tau)$ is rationally smooth. In particular, the constant sheaf on $S(\tau)$ is a semisimple complex.*

Proof. The trick is to reduce the proof to a problem in the Schubert varieties in $Y(\mathbb{C})$ and then apply the known results of their singular loci. Without loss of generality, we assume that $n = 1$. We only need to show that each connected component in $S(\tau)$ is rationally

smooth. Let us fix a connected component, say C , in $S(\tau)$. This is further reduced to show the rational smoothness of the inverse image, say C' , in $Y(\mathbb{C}) \times Y(\mathbb{C})$ of C under the projection from $Y(\mathbb{C}) \times Y(\mathbb{C})$ to a suitable connected component of ${}^iX(\mathbb{C}) \times {}^iX(\mathbb{C})$. Fix a flag F in $Y(\mathbb{C})$, and let $C'' = C' \cap \{F\} \times Y(\mathbb{C})$. It is finally reduced to show that C'' is rationally smooth. Observe that C'' is a Schubert variety in $Y(\mathbb{C})$. It is indexed by the permutation $\tau = (d-1, \dots, 1, \bar{d})$ in the Weyl group of type B_d in the notation in [BL00] and [B98]. This permutation does not contain any bad pattern in [BL00, 13.3.3], hence implies that C'' is rationally smooth (see [KL79, A1]) by [BL00, 8.3.16]. \square

A direct consequence of this lemma is that \mathbf{t} is a shadow of a semisimple complex. Its idempotent components are shadows of simple perverse sheaves, up to shifts.

5.4. Generators for \mathbf{S}^i . Recall \mathbf{S}^i is an \mathcal{A} -subalgebra of \mathbf{S}^j . Fix any $A \in {}^i\Xi_d$. By (3.25), $m_A \in \mathbf{S}^j$ can be generated by elements of the form $[D_{i,h,j} + a_{ij}E_{h+1,h}^\theta]$. Note that elements of the form $[D_{i,h,j} + a_{ij}E_{n,n+1}^\theta]$ or $[D_{i,h,j} + a_{ij}E_{n+1,n}^\theta]$ in general do not lie in \mathbf{S}^i . However, we have the following key observation:

(5.3) *The “twin product” $[D_{i,h,j} + a_{ij}E_{n,n+1}^\theta] * [D_{i,h,j} + a_{ij}E_{n+1,n}^\theta]$ is an element in \mathbf{S}^i . The monomial basis element m_A given in Theorem 3.10 is always a product of such twin products together with $[D_{i,h,j} + a_{ij}E_{h+1,h}^\theta] \in \mathbf{S}^i$ for $h \neq n, n+1$.*

Indeed, thanks to $A \in {}^i\Xi_d$, the $(n+1)$ -th entry of the row vector of $D_{i,h,j} + a_{ij}E_{n,n+1}^\theta$ is 1 and so $(D_{i,h,j})_{n+1,n+1} = 1$. By Proposition 3.7, for $R > 0$ and D with $D_{n+1,n+1} = 1$, we have

$$(5.4) \quad \begin{aligned} & [D + RE_{n,n+1}^\theta] * [D + RE_{n+1,n}^\theta] \\ &= [D + RE_{n,n+2}] + \sum_{i=1}^R v^{\beta(i)} \overline{\begin{bmatrix} D_{n,n} + i \\ i \end{bmatrix}} [D + iE_{n,n}^\theta + (R-i)E_{n,n+2}^\theta] \end{aligned}$$

where $\beta(i) = D_{n,n}i - i(i+1)/2$.

Thus we have $m_A \in \mathbf{S}^i$ for any $A \in {}^i\Xi_d$. For example, for $n = 2$, m_A is a product of 20 terms in (3.25) which simplifies to 14 terms thanks to $A \in {}^i\Xi_d$ as follows, and it actually lies in \mathbf{S}^i :

$$\begin{aligned} & [D_{2,1,1} + a_{21}E_{21}^\theta] * ([D_{4,3,1} + a_{41}E_{43}^\theta] * [D_{4,2,1} + a_{41}E_{32}^\theta]) * [D_{4,1,1} + a_{41}E_{21}^\theta] \\ & * ([D_{4,3,2} + a_{42}E_{43}^\theta] * [D_{4,2,2} + a_{42}E_{32}^\theta]) * [D_{5,4,1} + a_{51}E_{54}^\theta] \\ & * ([D_{5,3,1} + a_{51}E_{43}^\theta] * [D_{5,2,1} + a_{51}E_{32}^\theta]) * [D_{5,1,1} + a_{51}E_{21}^\theta] * [D_{5,4,2} + a_{52}E_{54}^\theta] \\ & * ([D_{5,3,2} + a_{52}E_{43}^\theta] * [D_{5,2,2} + a_{52}E_{32}^\theta]) * [D_{5,4,4} + a_{54}E_{54}^\theta]. \end{aligned}$$

We summarize the above discussions as follows.

Proposition 5.6. *For any $A \in {}^i\Xi_d$, we have $m_A \in \mathbf{S}^i$. Moreover, $\{m_A \mid A \in {}^i\Xi_d\}$ forms a monomial \mathcal{A} -basis for \mathbf{S}^i .*

Proposition 5.7. *The \mathcal{A} -algebra \mathbf{S}^i is generated by the elements $[D + RE_{n,n+2}^\theta]$, $[D + RE_{i,i+1}^\theta]$, $[D + RE_{i+1,i}^\theta]$ where $1 \leq i \leq n-1$, $R \in [0, d]$ and $D \in {}^i\Xi_{d-R}^{diag}$.*

By using Lemma 3.9, we have

$$[D + RE_{n,n}^\theta + E_{n,n+2}^\theta] * [D + E_{n,n}^\theta + RE_{n,n+2}^\theta] = [D + (R+1)E_{n,n+2}^\theta] + \text{lower terms}.$$

Hence Proposition 5.7 implies the following.

Corollary 5.8. *The $\mathbb{Q}(v)$ -algebra ${}_{\mathbb{Q}}\mathbf{S}^i$ is generated by the elements $[D']$ with $D' \in {}^i\Xi_d^{diag}$, and the elements $[D + E_{i,i+1}^\theta]$, $[D + E_{i+1,i}^\theta]$, $[D + E_{n,n+2}^\theta]$, where $1 \leq i \leq n-1$ and $D \in {}^i\Xi_{d-1}^{diag}$.*

Recall in §3.6, we have defined the canonical basis $\mathbf{B}_d^j = \{\{A\} \mid A \in \Xi_d\}$ for \mathbf{S}^j . The bar involution on \mathbf{S}^i is identified with the restriction from the bar involution on \mathbf{S}^j via the inclusion $\mathbf{S}^i \subset \mathbf{S}^j$ by the geometric construction. It follows from Proposition 5.6 (and recall every monomial basis element is bar invariant) that $\{A\} \in \mathbf{S}^i$, for $A \in {}^i\Xi_d$. In particular, we have the following.

Proposition 5.9. *The canonical basis for \mathbf{S}^i is given by $\mathbf{B}_d^j \cap \mathbf{S}^i = \{\{A\} \mid A \in {}^i\Xi_d\}$.*

Hence all the results on the inner product and canonical basis for \mathbf{S}^j in §3.6 and §3.7 make sense by restriction to the subalgebra \mathbf{S}^i . We also obtain a geometric realization of the second \imath Schur duality (on the Schur algebra level) as ${}_{\mathbb{Q}}\mathbf{S}^i \circledcirc {}_{\mathbb{Q}}\mathbf{T}_d^i \circledcirc {}_{\mathbb{Q}}\mathbf{H}_{B_d}$. To avoid much repetition, we will explain this duality in some detail in Appendix A, where the Schur algebra \mathbf{S}^i is replaced by a modified coideal algebra.

6. CONVOLUTION ALGEBRAS FROM GEOMETRY OF TYPE C

In this section, we formulate analogous constructions and results in type C. This could have been done in a completely analogous way as before, but to avoid much repetition we choose some short cuts to reduce the considerations quickly to the type B counterparts.

6.1. A first formulation. We fix the following data in this subsection:

- A pair of positive integers (n, d) such that $N = 2n$ and $D = 2d$ in Section 2.1;
- A non-degenerate skew-symmetric bilinear form $Q : \mathbb{F}_q^D \times \mathbb{F}_q^D \longrightarrow \mathbb{F}_q$.

Let $Sp(D)$ be the symplectic subgroup of $GL(D)$ consisting of all elements g such that $Q(gu, gu') = Q(u, u')$, for $u, u' \in \mathbb{F}_q^D$. We can define the sets $X_{\mathbf{C}_d}$, $Y_{\mathbf{C}_d}$, $\Xi_{\mathbf{C}_d}$, $\Pi_{\mathbf{C}_d}$ and $\Sigma_{\mathbf{C}_d}$ in formally the same way (in notations N and D) as the sets X , Y , Ξ_d , Π and Σ in Section 2.2, respectively. For example, $X_{\mathbf{C}_d}$ is the set of N -step flags $(V_i)_{0 \leq i \leq N}$ in \mathbb{F}_q^D satisfying $V_{N-i} = V_i^\perp$, and in particular, $V_n = V_n^\perp$ is Lagrangian.

We have the following analogue of Lemmas 2.1 and 2.2, whose proof is skipped.

Lemma 6.1. (a) *We have*

$$\#\Sigma_{\mathbf{C}_d} = 2^d \cdot d!, \quad \#\Pi_{\mathbf{C}_d} = (2n)^d, \quad \text{and} \quad \#\Xi_{\mathbf{C}_d} = \binom{2n^2 + d - 1}{d}.$$

(b) *We have canonical bijections $Sp(D) \backslash X_{\mathbf{C}_d} \times X_{\mathbf{C}_d} \leftrightarrow \Xi_{\mathbf{C}_d}$, $Sp(D) \backslash X_{\mathbf{C}_d} \times Y_{\mathbf{C}_d} \leftrightarrow \Pi_{\mathbf{C}_d}$, and $Sp(D) \backslash Y_{\mathbf{C}_d} \times Y_{\mathbf{C}_d} \leftrightarrow \Sigma_{\mathbf{C}_d}$.*

Following §2.3 we define

$${}^{\mathbf{C}}\mathbf{S}^i = \mathcal{A}_{Sp(D)}(X_{\mathbf{C}_d} \times X_{\mathbf{C}_d}), \quad {}^{\mathbf{C}}\mathbf{T}_d = \mathcal{A}_{Sp(D)}(X_{\mathbf{C}_d} \times Y_{\mathbf{C}_d}), \quad \mathbf{H}_{B_d} = \mathcal{A}_{Sp(D)}(Y_{\mathbf{C}_d} \times Y_{\mathbf{C}_d})$$

to be the space of $Sp(D)$ -invariant \mathcal{A} -valued functions on $X_{\mathbf{C}_d} \times X_{\mathbf{C}_d}$, $X_{\mathbf{C}_d} \times Y_{\mathbf{C}_d}$, and $Y_{\mathbf{C}_d} \times Y_{\mathbf{C}_d}$ respectively. (Note the superscript \imath here instead of j is used!) As before, ${}^{\mathbf{C}}\mathbf{S}^i$

is endowed with an \mathcal{A} -algebra structure via a convolution product. Also, the convolution algebra $\mathcal{A}_{Sp(D)}(Y_{\mathbf{C}_d} \times Y_{\mathbf{C}_d})$ is known to be canonically isomorphic to the Iwahori-Hecke algebra \mathbf{H}_{B_d} of type B_d , and so there is no ambiguity of notation above.

Associated to $B \in \Pi_{\mathbf{C}_d}$, we have a sequence of integers r_1, \dots, r_D as defined in (2.5) which satisfies the same bijections (2.6). We shall denote the characteristic function of the $Sp(D)$ -orbit \mathcal{O}_B by $e_{r_1 \dots r_d}$. The following is an analogue of Lemma 2.4, whose similar proof is skipped.

Lemma 6.2. *The right \mathbf{H}_{B_d} -action on ${}^{\mathbf{C}}\mathbf{T}_d$*

$${}^{\mathbf{C}}\mathbf{T}_d \times \mathbf{H}_{B_d} \longrightarrow {}^{\mathbf{C}}\mathbf{T}_d, \quad (e_{r_1 \dots r_d}, T_j) \mapsto e_{r_1 \dots r_d} T_j,$$

is given as follows. For $1 \leq j \leq d-1$, we have

$$(6.1) \quad e_{r_1 \dots r_d} T_j = \begin{cases} e_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j < r_{j+1}; \\ v^2 e_{r_1 \dots r_d}, & \text{if } r_j = r_{j+1}; \\ (v^2 - 1) e_{r_1 \dots r_d} + v^2 e_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j > r_{j+1}. \end{cases}$$

Moreover, (recalling $r_{d+1} = N + 1 - r_d$) we have

$$(6.2) \quad e_{r_1 \dots r_{d-1} r_d} T_d = \begin{cases} e_{r_1 \dots r_{d-1} r_{d+1}}, & \text{if } r_d < r_{d+1}; \\ v^2 e_{r_1 \dots r_{d-1} r_d}, & \text{if } r_d = r_{d+1}; \\ (v^2 - 1) e_{r_1 \dots r_{d-1} r_d} + v^2 e_{r_1 \dots r_{d-1} r_{d+1}}, & \text{if } r_d > r_{d+1}. \end{cases}$$

We set

$$\tilde{e}_{r_1 \dots r_d} = v^{\#\{(c, c') | c, c' \in [1, d+1], c < c', r_c < r_{c'}\}} e_{r_1 \dots r_d}.$$

The formulas (6.1) and (6.2) can be rewritten as follows. For $1 \leq j \leq d-1$,

$$(6.3) \quad \tilde{e}_{r_1 \dots r_d} T_j = \begin{cases} v \tilde{e}_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j < r_{j+1}; \\ v^2 \tilde{e}_{r_1 \dots r_d}, & \text{if } r_j = r_{j+1}; \\ (v^2 - 1) \tilde{e}_{r_1 \dots r_d} + v \tilde{e}_{r_1 \dots r_{j-1} r_{j+1} r_j r_{j+2} \dots r_d}, & \text{if } r_j > r_{j+1}; \end{cases}$$

$$(6.4) \quad \tilde{e}_{r_1 \dots r_{d-1} r_d} T_d = \begin{cases} v \tilde{e}_{r_1 \dots r_{d-1} r_{d+1}}, & \text{if } r_d < r_{d+1}; \\ v^2 \tilde{e}_{r_1 \dots r_{d-1} r_d}, & \text{if } r_d = r_{d+1}; \\ (v^2 - 1) \tilde{e}_{r_1 \dots r_{d-1} r_d} + v \tilde{e}_{r_1 \dots r_{d-1} r_{d+1}}, & \text{if } r_d > r_{d+1}. \end{cases}$$

Remark 6.3. The formulas (6.3) and (6.4) essentially coincide with the one given in [G97] in an opposite ordering. (Note that the presentation of Iwahori-Hecke algebras in [G97] is somewhat different from ours via T_i .) This shows that the \mathcal{A} -algebra ${}^{\mathbf{C}}\mathbf{S}^i$ is isomorphic to the hyperoctahedral Schur algebra in [G97].

6.2. A variation. We fix the following data in this subsection:

- A pair of positive integers (n, d) such that $N = 2n + 1$ and $D = 2d$ in Section 2.1;
- A non-degenerate skew-symmetric bilinear form $Q : \mathbb{F}_q^D \times \mathbb{F}_q^D \longrightarrow \mathbb{F}_q$.

(Note the only difference from the data in Section 6.1 is that $N = 2n + 1$ now.) We can define analogous sets as those in Section 6.1. Let $'X_{\mathbf{C}_d}$ be the set defined formally as $X_{\mathbf{C}_d}$ in Section 6.1 with now $N = 2n + 1$. We keep exactly the same $Y_{\mathbf{C}_d}$ as in Section 6.1.

Following §2.3, we can define an \mathcal{A} -algebra ${}^{\mathbf{C}}\mathbf{S}^j := \mathcal{A}_{Sp(D)}('X_{\mathbf{C}_d} \times 'X_{\mathbf{C}_d})$ with a convolution product; Similarly, we have a commuting left ${}^{\mathbf{C}}\mathbf{S}^j$ -action and a right \mathbf{H}_{B_d} -action on ${}^{\mathbf{C}}\mathbf{T}'_d := \mathcal{A}_{Sp(D)}('X_{\mathbf{C}_d} \times Y_{\mathbf{C}_d})$. As before (see (2.5) and (2.6)), ${}^{\mathbf{C}}\mathbf{T}'_d$ has a basis given by the characteristic functions $e_{r_1 \dots r_d}$, where the d -tuples $r_1 \dots r_d$ are in bijection with the $Sp(D)$ -orbits in $'X_{\mathbf{C}_d} \times Y_{\mathbf{C}_d}$.

There is a natural inclusion $X_{\mathbf{C}_d} \subset 'X_{\mathbf{C}_d}$, which identifies a flag $(\dots \subseteq V_n \subseteq \dots)$ with $(\dots \subseteq V_n \subseteq V_n \subseteq \dots)$.

Lemma 6.4. *The \mathcal{A} -algebra ${}^{\mathbf{C}}\mathbf{S}^i$ is naturally a subalgebra of ${}^{\mathbf{C}}\mathbf{S}^j$ induced by the inclusion $X_{\mathbf{C}_d} \subset 'X_{\mathbf{C}_d}$.*

The next lemma follows by similar arguments for Lemmas 2.2 and 2.4.

Lemma 6.5. (a) *We have*

$$\#(Sp(D) \backslash 'X_{\mathbf{C}_d} \times 'X_{\mathbf{C}_d}) = \binom{2n^2 + 2n + d}{d}, \quad \#(Sp(D) \backslash 'X_{\mathbf{C}_d} \times Y_{\mathbf{C}_d}) = (2n + 1)^d.$$

(b) *The right \mathbf{H}_{B_d} -action on ${}^{\mathbf{C}}\mathbf{T}'_d$ is given by the formulas (6.1)-(6.4).*

6.3. Type C vs type B. By Lemmas 2.2, 2.4, and 6.5, we have a right \mathbf{H}_{B_d} -module isomorphism ${}_{\mathbb{Q}}\mathbf{T}_d \xrightarrow{\sim} {}_{\mathbb{Q}}\mathbf{T}_d$ by sending $e_{r_1 \dots r_d}$ to the element in the same notation. This isomorphism induces an algebra isomorphism

$$\phi_1 : \text{End}_{\mathbf{H}_{B_d}}(\mathbf{T}_d) \xrightarrow{\sim} \text{End}_{\mathbf{H}_{B_d}}({}^{\mathbf{C}}\mathbf{T}'_d).$$

Earlier (see Proposition 4.12 and [P09]) we have obtained an \mathcal{A} -algebra isomorphism

$$\phi_2 : \mathbf{S}^j \longrightarrow \text{End}_{\mathbf{H}_{B_d}}(\mathbf{T}_d).$$

Similarly, we have an \mathcal{A} -algebra isomorphism

$$\phi_3 : {}^{\mathbf{C}}\mathbf{S}^j \longrightarrow \text{End}_{\mathbf{H}_{B_d}}({}^{\mathbf{C}}\mathbf{T}'_d).$$

The following proposition allows us to reduce the type C case to the type B case.

Proposition 6.6. *We have natural \mathcal{A} -algebra isomorphisms $\mathbf{S}^j \cong {}^{\mathbf{C}}\mathbf{S}^j$ and $\mathbf{S}^i \cong {}^{\mathbf{C}}\mathbf{S}^i$.*

Proof. The first isomorphism is given by $\phi_3^{-1}\phi_2\phi_1$, and the second isomorphism can be obtained similarly. \square

Actually, the above isomorphisms are canonical in the sense they match various bases; we treat one case below in some detail.

Proposition 6.7. *The isomorphism $\phi_3^{-1}\phi_2\phi_1 : \mathbf{S}^j \rightarrow {}^{\mathbf{C}}\mathbf{S}^j$ sends the basis element e_A , for $A \in \Xi_d$, to the element $e_{A-E_{n+1,n+1}}$.*

Proof. When A is a diagonal matrix, the statement holds by definition. When $A - E_{h,h\pm 1}^\theta$ is diagonal, the statement holds by checking directly that the action of $e_{A-E_{n+1,n+1}} \in {}^{\mathbf{C}}\mathbf{S}^j$ on $e_{r_1 \dots r_d}$ is the same as the action of $e_A \in \mathbf{S}^j$ on $e_{r_1 \dots r_d}$. Note that the counterpart of Lemma 3.2 for ${}^{\mathbf{C}}\mathbf{S}^j$ holds with the same structure constants. The proof is basically the same as that of Lemma 3.2 with $\epsilon_{n+1,n+1}^\theta$ defined to be 1 with the extra care that the number of isotropic lines in \mathbb{F}_q^D with respect to the skew-symmetric form is $\frac{q^D-1}{q-1}$. The proposition for general A follows from this by induction. \square

APPENDIX A. A GEOMETRIC SETTING FOR THE COIDEAL ALGEBRA $\dot{\mathbf{U}}^i$ AND
 COMPATIBILITY OF CANONICAL BASES
 (BY H. BAO, Y. LI, AND W. WANG)

In this Appendix, we construct an \mathcal{A} -algebra \mathbf{K}^i from the family of i Schur algebras \mathbf{S}^i as d varies. Since it is difficult to see directly the stabilization of \mathbf{S}^i whose new generators admit rather implicit multiplication formulas, we construct \mathbf{K}^i as a subalgebra of another limit algebra \mathbf{K}^j in a two-step process. We then show that \mathbf{K}^i is isomorphic to a subalgebra of a quotient of \mathbf{K}^j and also isomorphic to a quotient of a subalgebra of \mathbf{K}^j . All standard, monomial, and canonical bases of \mathbf{K}^i and \mathbf{K}^j (and the intermediate algebras) are shown to be compatible. We then show that \mathbf{K}^i is isomorphic to the modified coideal algebra $\dot{\mathbf{U}}^i$ associated to $\mathfrak{gl}(N-1)$, for $N = 2n+1$, and obtain a geometric realization of the $(\dot{\mathbf{U}}^i, \mathbf{H}_{B_d})$ -duality. Finally, we construct a surjective homomorphism $\phi_d^i : \mathbf{K}^i \rightarrow \mathbf{S}^i$ (which is compatible with the homomorphism $\phi_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$ in §4.6), and show that both homomorphisms ϕ_d and ϕ_d^i send canonical basis elements to canonical basis elements or zero.

A.1. A limit algebra \mathbf{K}^j and a subalgebra \mathbf{K}^i . We set $N = 2n+1$, and introduce the following subsets of $\tilde{\Xi}$:

$$\begin{aligned}\tilde{\Xi}_{<} &= \{A = (a_{ij}) \in \tilde{\Xi} \mid a_{n+1,n+1} < 0\}, \\ \tilde{\Xi}_{>} &= \{A = (a_{ij}) \in \tilde{\Xi} \mid a_{n+1,n+1} > 0\}.\end{aligned}$$

Recalling I is the identity $N \times N$ -matrix, for any $N \times N$ -matrix A and $p \in \mathbb{Z}$ we set

$$\check{I} = I - E_{n+1,n+1}, \quad \check{p}A = A + p\check{I}.$$

For $A \in \tilde{\Xi}_{>}$, we have $\check{p}A \in \Xi$ for $p \gg 0$.

Lemma A.1. *Given $A_1, A_2, \dots, A_f \in \tilde{\Xi}_{>}$, there exist $Z_i \in \tilde{\Xi}_{>}$ and $G_i(v, v') \in \mathbb{Q}(v)[v', v'^{-1}]$ ($1 \leq i \leq m$, for some m) such that*

$$(A.1) \quad [\check{p}A_1] * [\check{p}A_2] * \dots * [\check{p}A_f] = \sum_{i=1}^m G_i(v, v^{-p}) [\check{p}Z_i], \quad \text{for all even integers } p \gg 0.$$

Proof. Though the proof is entirely similar to the proof of [BLM, Proposition 4.2] (where ${}_pA = A + pI$ is used), we provide some details to assure a reader that the modification from ${}_pA$ to $\check{p}A$ does not cause extra problems. It suffices to prove the lemma for $f = 2$, as the general case follows by induction on f . We shall use Proposition 3.7 and the notations therein.

Let $1 \leq h \leq n$. We assume first that $A_1 - RE_{h,h+1}^\theta$ is diagonal. Let $A_2 = A = (a_{i,j})$. Let T be the set of all $t = (t_1, t_2, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{i=1}^N t_i = R$ and

$$\begin{cases} t_u \leq a_{h+1,u}, & \text{if } h < n \text{ and } u \neq h+1; \\ t_u + t_{N+1-u} \leq a_{h+1,u}, & \text{if } h = n. \end{cases}$$

For each $t \in T$, we define

$$G_t(v, v') = v^{\beta(t)} \prod_{\substack{u=1 \\ u \neq h}}^N \overline{\begin{bmatrix} a_{h,u} + t_u \\ t_u \end{bmatrix}} \prod_{i=1}^{t_h} \frac{v^{-2(a_{h,h} + t_h - i + 1)} v'^2 - 1}{v^{-2i} - 1} (v')^{-\delta_{h,n} \cdot \sum_{j < n+1} t_j}.$$

Then by Proposition 3.7, we have

$$[\check{p}A_1] * [\check{p}A] = \sum_{t \in T} G_t(v, v^{-p}) \left[\check{p} \left(A + \sum_{1 \leq u \leq N} t_u (E_{h,u}^\theta - E_{h+1,u}^\theta) \right) \right] \quad \text{for all even } p \gg 0.$$

Next we assume that $A_1 - RE_{h+1,h}^\theta$ is diagonal. Let $A_2 = A = (a_{i,j})$. Let T be the set of all $t = (t_1, t_2, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{i=1}^N t_i = R$ and $t_u \leq a_{h,u}$ for all $u \neq h$.

For each $t \in T$, if $h \neq n$, we define

$$G_t(v, v') = v^{\beta'(t)} \prod_{\substack{u=1 \\ u \neq h+1}}^N \overline{\left[\frac{a_{h+1,u} + t_u}{t_u} \right]}^{t_{h+1}} \prod_{i=1}^{t_{h+1}} \frac{v^{-2(a_{h+1,h+1} + t_{h+1} - i + 1)} (v')^2 - 1}{v^{-2i} - 1};$$

if $h = n$, we define

$$\begin{aligned} G_t(v, v') = & v^{\beta''(t)} \prod_{u < n+1} \overline{\left[\frac{a_{n+1,u} + t_u + t_{N+1-u}}{t_u} \right]} \prod_{u > n+1} \overline{\left[\frac{a_{n+1,u} + t_u}{t_u} \right]} \\ & \cdot \prod_{i=0}^{t_{n+1}-1} \overline{\left[\frac{a_{n+1,n+1} + 1 + 2i}{i+1} \right]} \cdot (v')^{\sum_{j \geq n+1} t_j}. \end{aligned}$$

By Proposition 3.7 we have

$$[\check{p}A_1] * [\check{p}A] = \sum_{t \in T} G_t(v, v^{-p}) \left[\check{p} \left(A + \sum_{1 \leq u \leq N} t_u (E_{h+1,u}^\theta - E_{h,u}^\theta) \right) \right] \quad \text{for even } p \gg 0.$$

Therefore we have proved so far that

$$[\check{p}A_1] * [\check{p}A_2] = \sum_{i=1}^m G_i(v, v^{-p}) [\check{p}Z_i] \quad \text{for even } p \gg 0,$$

where either $A_1 - RE_{h,h+1}^\theta$ or $A_1 - RE_{h+1,h}^\theta$ ($1 \leq h \leq n$) is diagonal. The proof of this formula for general $A_1 \in \tilde{\Xi}_>$ follows from the special cases which we have just established by the same induction as in [BLM, pp.668]. \square

We shall introduce an \mathcal{A} -algebra $\mathbf{K}_>^j$ below.

Corollary A.2. *Let $\mathbf{K}_>^j$ be a free \mathcal{A} -module with a basis $[A]$ for $A \in \tilde{\Xi}_>$. Then there is a unique structure of associative \mathcal{A} -algebra on $\mathbf{K}_>^j$ in which the product $[A_1] \cdot [A_2] \cdots [A_f]$ is given by $\sum_{i=1}^m G_i(v, 1) [Z_i]$ (in the notation of (A.1)).*

We remark the similarity of the stabilization procedure above leading to the algebra $\mathbf{K}_>^j$ with the one in type D given in [FL14].

Following the definition we obtain the following multiplication formula in the algebra $\mathbf{K}_>^j$. For any $A, B \in \tilde{\Xi}_>$ such that $\text{ro}(A) = \text{co}(B)$ and $B - RE_{h,h+1}^\theta$ being diagonal for some $h \in [1, n]$, we have

$$(A.2) \quad [B] \cdot [A] = \sum_t v^{\beta(t)} \prod_{u=1}^N \overline{\left[\frac{a_{hu} + t_u}{t_u} \right]} \left[A + \sum_{1 \leq u \leq N} t_u (E_{hu}^\theta - E_{h+1,u}^\theta) \right],$$

where $\beta(t)$ is defined in (3.18) and $t = (t_1, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^N t_u = R$ and $t_u \leq a_{h+1,u}$ for $u \neq h+1$ and $h < n$ or $t_u + t_{N+1-u} \leq a_{n+1,u}$ for $h = n$.

Similarly for any $A, C \in \tilde{\Xi}_>$ such that $\text{ro}(A) = \text{co}(C)$ and $C - RE_{h+1,h}^\theta$ being diagonal for some $h \in [1, n-1]$, we have

$$(A.3) \quad [C] \cdot [A] = \sum_t v^{\beta'(t)} \prod_{u=1}^N \overline{\begin{bmatrix} a_{h+1,u} + t_u \\ t_u \end{bmatrix}} \left[A - \sum_{1 \leq u \leq N} t_u (E_{hu}^\theta - E_{h+1,u}^\theta) \right],$$

where $\beta'(t)$ is defined in (3.19) and $t = (t_1, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^N t_u = R$ and $0 \leq t_u \leq a_{h,u}$ for $u \neq h$. For $h = n$, we have

$$(A.4) \quad [C] \cdot [A] = \sum_t v^{\beta''(t)} \prod_{u < n+1} \overline{\begin{bmatrix} a_{n+1,u} + t_u + t_{N+1-u} \\ t_u \end{bmatrix}} \prod_{u > n+1} \overline{\begin{bmatrix} a_{n+1,u} + t_u \\ t_u \end{bmatrix}} \\ \cdot \prod_{i=0}^{t_{n+1}-1} \frac{\overline{[a_{n+1,n+1} + 1 + 2i]}}{[i+1]} \left[A - \sum_{1 \leq u \leq N} t_u (E_{nu}^\theta - E_{n+1,u}^\theta) \right],$$

where $\beta''(t)$ is defined in (3.21) and $t = (t_1, \dots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^N t_u = R$ and $0 \leq t_u \leq a_{n,u}$ for $u \neq n$.

An entirely analogous argument as for the construction of the monomial basis for \mathbf{K}^j (see Theorem 3.10 and (4.8)), now using (A.2), (A.3) and (A.4), proves the following.

Proposition A.3. *The \mathcal{A} -algebra $\mathbf{K}_>^j$ admits a monomial basis $\{M_A \mid A \in \tilde{\Xi}_>\}$.*

In other words, the monomial basis element M_A , for $A \in \tilde{\Xi}_>$, is formally given by the same formula as in (4.8) though with a different multiplication.

Let \mathbf{K}^i be the \mathcal{A} -submodule of $\mathbf{K}_>^j$ generated by $[A]$, for $A \in {}^i\tilde{\Xi}$ (recall ${}^i\tilde{\Xi}$ from §5.1). Since any matrix $A \in \tilde{\Xi}_>$ with $\text{co}(A)_{n+1} = \text{ro}(A)_{n+1} = 1$ must lie in ${}^i\tilde{\Xi}$, we conclude from (A.2)–(A.4) that \mathbf{K}^i is a subalgebra of $\mathbf{K}_>^j$. By construction, the monomial basis of $\mathbf{K}_>^j$ restricts to a monomial basis $\{M_A \mid A \in {}^i\tilde{\Xi}\}$ for \mathbf{K}^i .

By a construction entirely analogous to [BLM, 4.3, 4.5], the \mathcal{A} -algebra $\mathbf{K}_>^j$ admits a natural bar involution $\overline{}$, which restricts to an involution on the subalgebra \mathbf{K}^i . Moreover, the bar involution satisfies the following property: for $A \in \tilde{\Xi}_>$ (respectively, $A \in {}^i\tilde{\Xi}$), we have

$$\overline{[A]} = [A] + \sum_{A' \sqsubset A} \tau_{A',A} [A'],$$

where A' runs over $\tilde{\Xi}_>$ (respectively, ${}^i\tilde{\Xi}$). Note $\overline{M_A} = M_A$ for all A . A standard argument like [Lu93, 24.2.1] implies the existence of a canonical basis $\{\{A\} \mid A \in \tilde{\Xi}_>\}$ for $\mathbf{K}_>^j$, which restrict to a canonical basis $\mathbf{B}^i := \{\{A\} \mid A \in {}^i\tilde{\Xi}\}$ for \mathbf{K}^i . (The defining properties of the canonical basis can be found in Proposition 4.2.)

Remark A.4. The \mathcal{A} -submodule H of \mathbf{K}^j spanned by $[A]$ for $A \in {}^i\tilde{\Xi}$ is not a subalgebra of \mathbf{K}^j (and thus we cannot define the algebra \mathbf{K}^i naively to be H though they have the same size). Indeed, one has the following example by using repeatedly Proposition 3.7 (for $a, b \in \mathbb{Z}$

with $b > 0$):

$$\begin{aligned}
& \begin{bmatrix} a+b-1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & a+b-1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{bmatrix} \\
&= v^{-a}(v^b - v^{-b}) \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{bmatrix} + v^b \overline{[b+1]} \begin{bmatrix} a-1 & 0 & b+1 \\ 0 & 1 & 0 \\ b+1 & 0 & a-1 \end{bmatrix} \\
&\quad + v^{b-1} \overline{[a+1]} \begin{bmatrix} a+1 & 0 & b-1 \\ 0 & 1 & 0 \\ b-1 & 0 & a+1 \end{bmatrix} + v^{-a+b-1}(1-v^{-2}) \begin{bmatrix} a & 1 & b-1 \\ 1 & -1 & 1 \\ b-1 & 1 & a \end{bmatrix}.
\end{aligned}$$

The last term on the right hand side of the above equation is clearly not in H , and hence H is not a subalgebra.

A.2. \mathbf{K}^i as a subquotient of \mathbf{K}^j . Let \mathcal{J} be the \mathcal{A} -submodule of \mathbf{K}^j spanned by $[A]$ for all $A \in \tilde{\Xi}_{<}$.

Lemma A.5. *The subspace \mathcal{J} is a two-sided ideal of \mathbf{K}^j .*

Proof. Assume we know for now that \mathcal{J} is a left ideal of \mathbf{K}^j . Since \mathcal{J} is clearly invariant under the anti-involution $[A] \mapsto v^{-d_A+d_{t_A}}[{}^t A]$, it follows that \mathcal{J} is also a right ideal of \mathbf{K}^j .

It remains to show that \mathcal{J} is a left ideal of \mathbf{K}^j . To that end it suffices to show that $[B] \cdot [A] \in \mathcal{J}$, for $A \in \tilde{\Xi}_{<}$ and $B \in \tilde{\Xi}$ such that $B - RE_{h,h+1}^\theta$ or $B - RE_{h+1,h}^\theta$ is diagonal, for some $1 \leq h \leq n$ and $R \geq 0$ (since such $[B]$ form a generating set of the algebra \mathbf{K}^j).

Let us first assume $B - RE_{h,h+1}^\theta$ is diagonal. Then by the multiplication formula in Proposition 3.7(a), the terms arising in $[B] \cdot [A]$ are $[A + \sum_u t_u(E_{hu}^\theta - E_{h+1,u}^\theta)]$, where $t_u \geq 0$. Clearly the $(n+1, n+1)$ -entry of $[A + \sum_u t_u(E_{hu}^\theta - E_{h+1,u}^\theta)]$ does not exceed $a_{n+1,n+1}$ (and recall $a_{n+1,n+1} < 0$ by assumption $A \in \tilde{\Xi}_{<}$). Hence we have $[A + \sum_u t_u(E_{hu}^\theta - E_{h+1,u}^\theta)] \in \mathcal{J}$ and $[B] \cdot [A] \in \mathcal{J}$.

Now assume $B - RE_{h+1,h}^\theta$ is diagonal. The same argument as above using Proposition 3.7(b) shows that $[B] \cdot [A] \in \mathcal{J}$ unless $h = n$. If $h = n$, by Proposition 3.7(b) (where C is replaced by B here) we have

$$\begin{aligned}
(A.5) \quad [B] \cdot [A] &= \sum_t v^{\beta''(t)} \prod_{u < n+1} \overline{\begin{bmatrix} a_{n+1,u} + t_u + t_{N+1-u} \\ t_u \end{bmatrix}} \prod_{u > n+1} \overline{\begin{bmatrix} a_{n+1,u} + t_u \\ t_u \end{bmatrix}} \\
&\quad \cdot \prod_{i=0}^{t_{n+1}-1} \overline{\frac{[a_{n+1,n+1} + 1 + 2i]}{[i+1]}} \left[A - \sum_{u=1}^N t_u(E_{nu}^\theta - E_{n+1,u}^\theta) \right].
\end{aligned}$$

The $(n+1, n+1)$ -entry of $\left[A - \sum_{u=1}^N t_u(E_{nu}^\theta - E_{n+1,u}^\theta) \right]$ is $a_{n+1,n+1} + 2t_{n+1}$ (which is an odd integer). If $a_{n+1,n+1} + 2t_{n+1} > 0$, then the coefficient of the term $\left[A - \sum_{u=1}^N t_u(E_{nu}^\theta - E_{n+1,u}^\theta) \right]$ must be zero since (here we recall $a_{n+1,n+1} < 0$)

$$\prod_{i=0}^{t_{n+1}-1} \overline{\frac{[a_{n+1,n+1} + 1 + 2i]}{[i+1]}} = 0.$$

Therefore, by (A.5), we have $[B] \cdot [A] \in \mathcal{J}$.

The lemma is proved. \square

Recall the monomial basis $\{\mathbf{M}_A \mid A \in \tilde{\Xi}\}$ of \mathbf{K}^j defined in (4.8).

Lemma A.6. *For $A \in \tilde{\Xi}_{<}$, we have*

- (1) $\mathbf{M}_A \in \mathcal{J}$;
- (2) $\mathbf{M}_A = [A] + \sum_{\substack{A' \in \tilde{\Xi}_{<} \\ A' \sqsubset A}} T_{A,A'} [A']$, for $T_{A,A'} \in \mathcal{A}$.

Hence \mathcal{J} admits a monomial basis $\{\mathbf{M}_A \mid A \in \tilde{\Xi}_{<}\}$, and the quotient \mathcal{A} -algebra \mathbf{K}^j/\mathcal{J} admits a monomial basis $\{\mathbf{M}_A + \mathcal{J} \mid A \in \tilde{\Xi}_{>}\}$.

Proof. Let us write $\mathbf{M}_A = [X_1] \cdot [X_2] \cdot \dots \cdot [X_m]$, where $[X_i]$ denote the divided power factors in (4.8). By construction we have

$$(A.6) \quad [X_r] \cdot \dots \cdot [X_m] = [A_r] + \sum_{A'_r \sqsubset A_r} C_{A_r, A'_r} [A'_r] = \mathbf{M}_{A_r},$$

for $A_r \in \tilde{\Xi}$ and $1 \leq r \leq m$. The monomial \mathbf{M}_A is constructed in a way such that we fill the entries of A (along the sequence of matrices $X_m = A_m, A_{m-1}, \dots, A_1 = A$) below the diagonal from bottom to top and from right to left. Therefore, when we have filled exactly the lower-triangular (i, j) -th entries with $n+1 < i$ and $j < i$ the resulting matrix $A_\ell = (a_{ij}^\ell)$, for some unique ℓ , satisfies

$$(A.7) \quad a_{ij}^\ell = \begin{cases} 0, & \text{if } j < i \leq n+1; \\ a_{ij}, & \text{if } n+1 < i \text{ and } j < i. \end{cases}$$

Recall we have $a_{n+1, n+1} < 0$ by the assumption that $A \in \tilde{\Xi}_{<}$. By (A.6), we have $\text{co}(X_m)_{n+1} = \text{co}(A)_{n+1} = \text{co}(A_r)_{n+1}$ (for all r , in particular for $r = \ell$), and thus

$$\begin{aligned} a_{n+1, n+1}^\ell &= \text{co}(A_\ell)_{n+1} - 2 \sum_{i > n+1} a_{i, n+1}^\ell \\ &= \text{co}(A)_{n+1} - 2 \sum_{i > n+1} a_{i, n+1} = a_{n+1, n+1} < 0. \end{aligned}$$

Hence it follows by (A.6) and (A.7) that

$$\text{ro}(X_\ell)_{n+1} = \text{ro}(A_\ell)_{n+1} = a_{n+1, n+1}^\ell < 0.$$

Since all entries in the $(n+1)$ st row of X_ℓ except the diagonal one $(X_\ell)_{n+1, n+1}$ are non-negative, we conclude that $(X_\ell)_{n+1, n+1} < 0$, i.e., $X_\ell \in \tilde{\Xi}_{<}$. It follows by Lemma A.5 that $\mathbf{M}_A = [X_1] \cdot \dots \cdot [X_m] \in \mathcal{J}$. This proves (1). The remaining statements follow from this. \square

The following can be rephrased in the terminology of Lusztig [Lu93] that \mathcal{J} is a based submodule (or ideal) of \mathbf{K}^j .

Proposition A.7. (1) *The ideal \mathcal{J} admits a canonical basis $\mathbf{B}^j \cap \mathcal{J} = \{\{A\} \mid A \in \tilde{\Xi}_{<}\}$.*
 (2) *The quotient \mathcal{A} -algebra \mathbf{K}^j/\mathcal{J} admits a canonical basis $\{\{A\} + \mathcal{J} \mid A \in \tilde{\Xi}_{>}\}$.*

Proof. It follows by Lemma A.6 that \mathcal{J} is a bar invariant submodule of \mathbf{K}^j and \mathcal{J} admits a canonical basis C_A for $A \in \tilde{\Xi}_{<}$, where C_A satisfies $\overline{C_A} = C_A$ and $C_A \in [A] + \sum_{A' \sqsubset A} v^{-1} \mathbb{Q}[v^{-1}][A']$. Since $C_A \in \mathcal{J} \subset \mathbf{K}^j$ and the canonical basis element $\{A\}$ satisfies the same characterization, it follows by uniqueness that $C_A = \{A\}$. This proves (1), and (2) follows directly from (1). \square

The precise relation between $\mathbf{K}^j_{>}$ and \mathbf{K}^j is given as follows.

Proposition A.8. *We have an isomorphism of \mathcal{A} -algebras $\sharp : \mathbf{K}^j/\mathcal{J} \longrightarrow \mathbf{K}^j_{>}$, which sends $[A] + \mathcal{J} \longmapsto [A]$, for $A \in \tilde{\Xi}_{>}$. Moreover, the isomorphism \sharp matches the corresponding monomial bases and canonical bases.*

Proof. We shall denote $[A]' = [A] + \mathcal{J}$ in this proof. It is clear that \sharp is a linear isomorphism. It remains to prove \sharp is an algebra homomorphism, i.e., the structure constants with respect to the standard basis are the same. Let us consider the products

$$[B]' \cdot [A]' = \sum_C f(C) [C]' \in \mathbf{K}^j/\mathcal{J} \quad \text{and} \quad [B] \cdot [A] = \sum_C g(C) [C] \in \mathbf{K}^j_{>},$$

for $B, A \in \tilde{\Xi}_{>}$. If one of $B - RE_{h,h+1}^\theta$ and $B - RE_{h+1,h}^\theta$ is diagonal, the identity $f(C) = g(C)$ follows by comparing the corresponding multiplication formulas (4.5)–(4.7) and (A.2)–(A.4). This implies that \sharp is an algebra isomorphism since $[B]'$ (and respectively, $[B]$) with B satisfying these conditions are generators of the algebra \mathbf{K}^j/\mathcal{J} (and respectively, $\mathbf{K}^j_{>}$).

Since \sharp matches the Chevalley generators, \sharp matches the monomial bases (which are given by the same ordered products of the generators), and \sharp commutes with the bar involutions. Finally, \sharp matches the canonical bases as partial orders \sqsubseteq are compatible. \square

We record the following identity in $\mathbf{K}^j/\mathcal{J} (\cong \mathbf{K}^j_{>})$, which follows from (A.2):

$$(A.8) \quad [D + E_{n,n+1}^\theta] \cdot [D + E_{n+1,n}^\theta] = [D + E_{n,n+2}^\theta] + v^{D_{n,n}-1} \overline{[D_{n,n} + 1]} [D + E_{n,n}^\theta].$$

Combining the results on bases on $\mathbf{K}^j_{>}$ and \mathbf{K}^i with Proposition A.8, we have established the following relation between \mathbf{K}^i and \mathbf{K}^j .

Theorem A.9. *The \mathcal{A} -algebra \mathbf{K}^i is naturally isomorphic to a subquotient of the \mathcal{A} -algebra \mathbf{K}^j (that is, a subalgebra of the quotient \mathbf{K}^j/\mathcal{J}), with compatible standard, monomial, and canonical bases.*

By abuse of notation, we will continue to use $[A], \mathbf{M}_A$ and $\{A\}$ (for $A \in {}^i\tilde{\Xi}$) to denote the elements of standard, monomial and canonical bases of \mathbf{K}^i .

A.3. Another definition of \mathbf{K}^i . Let \mathbf{K}_1^j be the \mathcal{A} -submodule of \mathbf{K}^j spanned by $[A]$ where $A \in \tilde{\Xi}$ satisfies that $\text{ro}(A)_{n+1} = \text{co}(A)_{n+1} = 1$. Then \mathbf{K}_1^j is naturally a subalgebra of \mathbf{K}^j . One can reformulate that $\mathbf{K}_1^j = \oplus_{\lambda, \mu} [D] \mathbf{K}^j [D']$ for various diagonal matrices D, D' whose $(n+1, n+1)$ -entries are 1 (recall $[D]$ are idempotents). Thus clearly the monomial and canonical bases of \mathbf{K}^j restricts to the monomial and canonical bases of \mathbf{K}_1^j .

By Lemma A.5, $\mathcal{J}_1 := \mathcal{J} \cap \mathbf{K}_1^j$ is a (two-sided) ideal of \mathbf{K}_1^j ; more explicitly we have

$$\mathcal{J}_1 = \mathcal{A}\text{-span}\{[A] \mid \text{ro}(A)_{n+1} = \text{co}(A)_{n+1} = 1, a_{n+1,n+1} < 0\}.$$

Proposition A.10. (1) *The monomial and canonical bases of \mathbf{K}^j (or of \mathbf{K}_1^j) restrict to the monomial and canonical bases of \mathcal{J}_1 .*

- (2) The quotient \mathcal{A} -algebra $\mathbf{K}_1^j/\mathcal{J}_1$ has a monomial basis $\{M_A + \mathcal{J}_1 \mid A \in {}^v\tilde{\Xi}\}$ and a canonical basis $\{\{A\} + \mathcal{J}_1 \mid A \in {}^v\tilde{\Xi}\}$.

Proof. It is a direct consequence of Lemma A.6 and Proposition A.7. \square

The composition of the inclusion and the quotient homomorphism $\mathbf{K}_1^j \rightarrow \mathbf{K}_>^j \rightarrow \mathbf{K}_>^j/\mathcal{J}$ has kernel $\mathcal{J} \cap \mathbf{K}_1^j = \mathcal{I}$. This induces an embedding of \mathcal{A} -algebras $\mathbf{K}_1^j/\mathcal{J}_1 \rightarrow \mathbf{K}_>^j/\mathcal{J}$, whose image is identified as \mathbf{K}^v by Proposition A.8. Hence we have obtained an isomorphism $\chi : \mathbf{K}_1^j/\mathcal{J}_1 \xrightarrow{\cong} \mathbf{K}^v$, which preserves the standard basis as well as the monomial basis. Since χ clearly commutes with the bar involutions and the partial orders \sqsubseteq on both algebras are compatible, χ also preserves the canonical basis. Summarizing, we have proved the following.

Proposition A.11. *We have an \mathcal{A} -algebra isomorphism $\chi : \mathbf{K}_1^j/\mathcal{J}_1 \cong \mathbf{K}^v$, which matches the corresponding standard, monomial, and canonical bases (that are all parametrized by ${}^v\tilde{\Xi}$).*

Hence, \mathbf{K}^v can be viewed naturally as a subquotient of \mathbf{K}^j in two different and complementary ways (see Theorem A.9 and Proposition A.11). The constructions so far in this Appendix can be summarized in the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{K}_1^j/\mathcal{J}_1 & \longleftarrow & \mathbf{K}_1^j & \longrightarrow & \mathbf{K}^j \\ \cong \downarrow \chi & & \downarrow & & \downarrow \\ \mathbf{K}^v & \longrightarrow & \mathbf{K}_<^j & \xleftarrow[\#]{\cong} & \mathbf{K}^j/\mathcal{J} \end{array}$$

Denote ${}_{\mathbb{Q}}\mathbf{K}^v = \mathbb{Q}(v) \otimes_{\mathcal{A}} \mathbf{K}^v$. Similar arguments as for the Schur algebra \mathbf{S}^v (see Proposition 5.7 and Corollary 5.8) give us the following.

Corollary A.12. (a) *The \mathcal{A} -algebra \mathbf{K}^v is generated by the elements $[D + RE_{n,n+2}^\theta]$, $[D + RE_{i,i+1}^\theta]$, $[D + RE_{i+1,i}^\theta]$, for $1 \leq i \leq n-1$, $R \in \mathbb{N}$, and $D \in {}^v\tilde{\Xi}^{diag}$.*
 (b) *The $\mathbb{Q}(v)$ -algebra ${}_{\mathbb{Q}}\mathbf{K}^v$ is generated by $[D]$, $[D + E_{n,n+2}^\theta]$, $[D + E_{i,i+1}^\theta]$, $[D + E_{i+1,i}^\theta]$, for $1 \leq i \leq n-1$ and $D \in {}^v\tilde{\Xi}^{diag}$.*

Based on the formulas (5.1), (5.2) and (5.4), we introduce the following elements in \mathbf{K}^v :

$$\begin{aligned} \mathbf{t}[D_\lambda] &:= [D_\lambda - E_{n,n}^\theta + E_{n,n+2}^\theta] + v^{-\lambda_n}[D_\lambda] \\ (A.9) \quad &= [D_\lambda - E_{n,n}^\theta + E_{n+1,n}^\theta] \cdot [D_\lambda - E_{n,n}^\theta + E_{n,n+1}^\theta] - [\lambda_n - \lambda_{n+1}][D_\lambda]. \end{aligned}$$

Lemma A.13. *The following identity holds in the algebra \mathbf{S}^v (and respectively, \mathbf{K}^v): for $A \in {}^v\Xi_d$ (and respectively, ${}^v\tilde{\Xi}$),*

$$\mathbf{t}[D_{\text{ro}(A)}] * [A] = \sum_i v^{\sum_{i \geq l} a_{n+2,l} - \sum_{i > l} a_{n,l} - \sum_{l > n+1} \delta_{i,l}} \overline{[a_{n+2,i} + 1]} [A - E_{n,i}^\theta + E_{n+2,i}^\theta],$$

where the summation is taken over $1 \leq i \leq N$ such that $A - E_{n,i}^\theta + E_{n+2,i}^\theta \in {}^v\Xi_d$ (and respectively, ${}^v\tilde{\Xi}$).

Proof. We will only give the detail in the setting of \mathbf{S}^v , and the other case is similar. It is understood that $[X] = 0$ in the proof below for X with any negative entry. Recalling

$a_{n+1,l} = \delta_{n+1,l}$, $\mathbf{S}^i \subset \mathbf{S}^j$, and using the multiplication formulas in \mathbf{S}^j , we have

$$\begin{aligned}
(A.10) \quad & [D_{\text{ro}(A)} - E_{n,n}^\theta + E_{n,n+1}^\theta] * [D_{\text{ro}(A)} - E_{n,n}^\theta + E_{n+1,n}^\theta] * [A] \\
&= \sum_{1 \leq i \leq N, i \neq n+1} v^{\sum_{i \geq l} a_{n+1,l} - \sum_{i > l} a_{n,l} + \sum_{N+1-i \leq l} a_{n,l} - 2\delta_{i > n+1}} \overline{[a_{n,N+1-i} + 1]} [A - E_{n,i}^\theta + E_{n+2,i}^\theta] \\
&\quad + \left(\sum_{1 \leq i \leq N} v^{\sum_{i \geq l} a_{n+1,l} - \sum_{i > l} a_{n,l} + \sum_{i \leq l} a_{n,l} - \delta_{i < n+1} - 1} \overline{[a_{n,i}]} \right) [A] \\
&= \sum_{1 \leq i \leq N, i \neq n+1} v^{\sum_{i \leq l} a_{n+2,l} - \sum_{i < l} a_{n,l} - \delta_{i > n+1}} \overline{[a_{n+2,i} + 1]} [A - E_{n,i}^\theta + E_{n+2,i}^\theta] \\
&\quad + \left(\sum_{1 \leq i \leq N} v^{\sum_{i \leq l} a_{n,l} - \sum_{i > l} a_{n,l} + \delta_{i > n+1} - \delta_{i < n+1} - 1} \overline{[a_{n,i}]} \right) [A].
\end{aligned}$$

Observe that

$$v^{-\sum_{i > l} a_{n,l} + \sum_{i \leq l} a_{n,l}} \overline{[a_{n,i}]} = \frac{v^{-\sum_{i \geq l} a_{n,l} + \sum_{i < l} a_{n,l}} - v^{-\sum_{i > l} a_{n,l} + \sum_{i \leq l} a_{n,l}}}{v^{-2} - 1}.$$

This leads to the following simplifications:

$$\begin{aligned}
\sum_{i > n+1} v^{-\sum_{i > l} a_{n,l} + \sum_{i \leq l} a_{n,l}} \overline{[a_{n,i}]} &= \frac{-v^{-\sum_{n+2 > l} a_{n,l} + \sum_{n+2 \leq l} a_{n,l}} + v^{-\text{ro}(A)_n}}{v^{-2} - 1}, \\
\sum_{i < n+1} v^{-\sum_{i > l} a_{n,l} + \sum_{i \leq l} a_{n,l} - 2} \overline{[a_{n,i}]} &= \frac{-v^{\text{ro}(A)_n - 2} + v^{-\sum_{n \geq l} a_{n,l} + \sum_{n < l} a_{n,l} - 2}}{v^{-2} - 1}.
\end{aligned}$$

By adding the above two equations while keeping in mind again that $a_{n+1,l} = \delta_{n+1,l}$, we rewrite the coefficient of $[A]$ on the right-hand side of (A.10) as

$$\sum_{1 \leq i \leq N} v^{\sum_{i \leq l} a_{n,l} - \sum_{i > l} a_{n,l} + \delta_{i > n+1} - \delta_{i < n+1} - 1} \overline{[a_{n,i}]} = [\text{ro}(A)_n - 1] + v^{\sum_{n+1 \geq l} a_{n+2,l} - \sum_{n+1 > l} a_{n,l}}.$$

The lemma now follows from (A.9) and the identity (A.10). \square

Denote by \mathbf{T} the $\mathbb{Q}(v)$ -subalgebra of ${}_{\mathbb{Q}}\mathbf{K}^i$ generated by $[D_\lambda]$ and $\mathbf{t}[D_\lambda]$ for $\lambda \in {}^i\tilde{\Xi}^{\text{diag}}$.

Corollary A.14. *For $k \geq 1$ and $\lambda \in {}^i\tilde{\Xi}^{\text{diag}}$, $\mathbf{t}^k[D_\lambda] = [k]![D_{\lambda^k} + kE_{n,n+2}^\theta] +$ a linear combination of $[D_{\lambda^r} + rE_{n,n+2}^\theta]$, for some D_{λ^r} and $0 \leq r \leq k-1$. In particular, the $\mathbb{Q}(v)$ -algebra \mathbf{T} has a basis $\{[D_\lambda + rE_{n,n+2}^\theta] \mid \forall r \in \mathbb{Z}_{\geq 0}, \lambda \in {}^i\tilde{\Xi}^{\text{diag}}\}$.*

Proof. The first statement follows by an easy induction on $k \geq 1$ via Lemma A.13. Note that the base case of the induction is the formula (A.9). The second statement follows from the first one since the linear independence of the elements $[D_\lambda + rE_{n,n+2}^\theta]$ in \mathbf{K}^i is clear by definition. \square

A.4. Isomorphism ${}_{\mathcal{A}}\dot{\mathbf{U}}^i \cong \mathbf{K}^i$. Below we formulate the counterparts of Sections 4.3–4.7. The proofs are very similar and hence will be often omitted.

The algebra \mathbf{U}^i is defined to be the associative algebra over $\mathbb{Q}(v)$ generated by $e_i, f_i, d_a, d_a^{-1}, t$, for $i = 1, 2, \dots, n-1$ and $a = 1, 2, \dots, n$, subject to the following relations:

for $i, j = 1, 2, \dots, n-1$, $a, b = 1, 2, \dots, n$,

$$(A.11) \quad \left\{ \begin{array}{ll} d_a d_a^{-1} = d_a^{-1} d_a = 1, \\ d_a d_b = d_b d_a, \\ d_a e_j d_a^{-1} = v^{-\delta_{a,j+1} - \delta_{N+1-a,j+1} + \delta_{a,j}} e_j, \\ d_a f_j d_a^{-1} = v^{-\delta_{a,j} + \delta_{a,j+1} + \delta_{N+1-a,j+1}} f_j, \\ e_i f_j - f_j e_i = \delta_{i,j} \frac{d_i d_{i+1}^{-1} - d_i^{-1} d_{i+1}}{v - v^{-1}}, \\ e_i^2 e_j + e_j e_i^2 = \llbracket 2 \rrbracket e_i e_j e_i, & \text{if } |i - j| = 1, \\ f_i^2 f_j + f_j f_i^2 = \llbracket 2 \rrbracket f_i f_j f_i, & \text{if } |i - j| = 1, \\ e_i e_j = e_j e_i, & \text{if } |i - j| > 1, \\ f_i f_j = f_j f_i, & \text{if } |i - j| > 1, \\ f_i t = t f_i, & \text{if } i \neq n-1, \\ t^2 f_{n-1} + f_{n-1} t^2 = \llbracket 2 \rrbracket t f_{n-1} t + f_{n-1}, \\ f_{n-1}^2 t + t f_{n-1}^2 = \llbracket 2 \rrbracket f_{n-1} t f_{n-1}, \\ e_i t = t e_i, & \text{if } i \neq n-1, \\ t^2 e_{n-1} + e_{n-1} t^2 = \llbracket 2 \rrbracket t e_{n-1} t + e_{n-1}, \\ e_{n-1}^2 t + t e_{n-1}^2 = \llbracket 2 \rrbracket e_{n-1} t e_{n-1}. \end{array} \right.$$

This algebra is a coideal subalgebra of $\mathbf{U}(\mathfrak{gl}(N-1))$, and hence it is a \mathfrak{gl} -version of the coideal algebra in the same notation defined in [BW13, §2.1] (which is a coideal subalgebra of $\mathbf{U}(\mathfrak{sl}(N+1))$).

Similar to the construction of $\dot{\mathbf{U}}^j$ in §4.3, we can define the modified quantum algebra $\dot{\mathbf{U}}^i$ for \mathbf{U}^i , where the unit of \mathbf{U}^i is replaced by a collection of orthogonal idempotents D_λ for $\lambda \in {}^i\tilde{\Xi}^{\text{diag}}$. Moreover, $\dot{\mathbf{U}}^i$ is naturally a \mathbf{U}^i -bimodule. By introducing similarly ${}_\lambda \mathbf{U}_{\lambda'}^i$, for $\lambda, \lambda' \in {}^i\tilde{\Xi}^{\text{diag}}$, we have

$$\begin{aligned} \dot{\mathbf{U}}^i &= \bigoplus_{\lambda, \lambda' \in {}^i\tilde{\Xi}^{\text{diag}}} {}_\lambda \mathbf{U}_{\lambda'}^i \\ &= \sum_{\lambda \in {}^i\tilde{\Xi}^{\text{diag}}} \mathbf{U}^i D_\lambda = \sum_{\lambda \in {}^i\tilde{\Xi}^{\text{diag}}} D_\lambda \mathbf{U}^i. \end{aligned}$$

In the same way establishing the presentation for ${}_{\mathbb{Q}}\dot{\mathbf{U}}^j$ given in Proposition 4.6, we can show that the $\mathbb{Q}(v)$ -algebra ${}_{\mathbb{Q}}\dot{\mathbf{U}}^i$ is isomorphic to the $\mathbb{Q}(v)$ -algebra generated by the symbols D_λ , $e_i D_\lambda$, $D_\lambda e_i$, $f_i D_\lambda$, $D_\lambda f_i$, $t D_\lambda$, and $D_\lambda t$, for $i = 1, \dots, n-1$ and $\lambda \in {}^i\tilde{\Xi}^{\text{diag}}$, subject to the following relations (A.12):

for $i, j = 1, \dots, n-1$, $\lambda, \lambda' \in {}^i\tilde{\Xi}^{\text{diag}}$, and for $x, x' \in \{1, e_i, e_j, f_i, f_j, t\}$,

$$(A.12) \left\{ \begin{array}{ll} xD_\lambda D_{\lambda'} x' &= \delta_{\lambda, \lambda'} xD_\lambda x', \\ e_i D_\lambda &= D_{\lambda - \alpha_i} e_i, \\ f_i D_\lambda &= D_{\lambda + \alpha_i} f_i, \\ tD_\lambda &= D_\lambda t, \\ e_i D_\lambda f_j &= f_j D_{\lambda - \alpha_i - \alpha_j} e_i, & \text{if } i \neq j, \\ e_i D_\lambda f_i &= f_i D_{\lambda - 2\alpha_i} e_i + [\lambda_{i+1} - \lambda_i] D_{\lambda - \alpha_i}, \\ (e_i^2 e_j + e_j e_i^2) D_\lambda &= [2] e_i e_j e_i D_\lambda, & \text{if } |i - j| = 1, \\ (f_i^2 f_j + f_j f_i^2) D_\lambda &= [2] f_i f_j f_i D_\lambda, & \text{if } |i - j| = 1, \\ e_i e_j D_\lambda &= e_j e_i D_\lambda, & \text{if } |i - j| > 1, \\ f_i f_j D_\lambda &= f_j f_i D_\lambda, & \text{if } |i - j| > 1, \\ t f_i D_\lambda &= f_i t D_\lambda, & \text{if } i \neq n-1, \\ (t^2 f_{n-1} + f_{n-1} t^2) D_\lambda &= ([2] t f_{n-1} t + f_{n-1}) D_\lambda, \\ (f_{n-1}^2 t + t f_{n-1}^2) D_\lambda &= [2] f_{n-1} t f_{n-1} D_\lambda, \\ t e_i D_\lambda &= e_i t D_\lambda, & \text{if } i \neq n-1, \\ (t^2 e_{n-1} + e_{n-1} t^2) D_\lambda &= ([2] t e_{n-1} t + e_{n-1}) D_\lambda, \\ (e_{n-1}^2 t + t e_{n-1}^2) D_\lambda &= [2] e_{n-1} t e_{n-1} D_\lambda. \end{array} \right.$$

To simplify the notation, we shall write $x_1 D_{\lambda^1} \cdot x_2 D_{\lambda^2} \cdots x_l D_{\lambda^l} = x_1 x_2 \cdots x_l D_{\lambda^l}$, if the product is not zero.

Theorem A.15. *We have an isomorphism of $\mathbb{Q}(v)$ -algebras $\aleph^i : \dot{\mathbf{U}}^i \rightarrow {}_{\mathbb{Q}}\mathbf{K}^i$ which sends*

$$\begin{aligned} D_\lambda &\mapsto [D_\lambda], & tD_\lambda &\mapsto [D_\lambda - E_{n,n}^\theta + E_{n,n+2}^\theta] + v^{-\lambda_n} [D_\lambda], \\ e_i D_\lambda &\mapsto [D_\lambda - E_{i,i}^\theta + E_{i+1,i}^\theta], & f_i D_\lambda &\mapsto [D_\lambda - E_{i+1,i+1}^\theta + E_{i,i+1}^\theta], \end{aligned}$$

for all $1 \leq i \leq n-1$ and $\lambda \in {}^i\tilde{\Xi}^{\text{diag}}$.

Proof. We first remark that the somewhat unusual formula for tD_λ has its origin in the formulas (5.1)-(5.2); by the notation (A.9), $\aleph^i(tD_\lambda) = \mathbf{t}[D_\lambda]$.

Via a direct computation one can check that \aleph^i is an algebra homomorphism by using the multiplication formulas. We illustrate here by checking that the map \aleph^i preserves one of the most complicated Serre-type relations, that is,

$$(A.13) \quad \aleph^i((t^2 f_{n-1} + f_{n-1} t^2) D_\lambda) = \aleph^i([2] t f_{n-1} t + f_{n-1}) D_\lambda.$$

The first term on the left hand side of (A.13) is

$$\begin{aligned} &\aleph^i(t^2 f_{n-1} D_\lambda) \\ &= v[2][D_\lambda - 3E_{n,n}^\theta + 2E_{n+2,n}^\theta + E_{n-1,n}^\theta] + [\lambda_n - 1][D_\lambda - E_{n,n}^\theta + E_{n-1,n}^\theta] \\ &\quad + v^{-\lambda_n+3}[D_\lambda - 2E_{n,n}^\theta + E_{n,n+2}^\theta + E_{n-1,n}^\theta] \\ &\quad + v^{-\lambda_n+1}[D_\lambda - 2E_{n,n}^\theta + E_{n,n+2}^\theta + E_{n-1,n}^\theta] + v^{-2\lambda_n+2}[D_\lambda - E_{n,n}^\theta + E_{n-1,n}^\theta]. \end{aligned}$$

The second term on the left hand side of (A.13) is

$$\begin{aligned} & \aleph^i(f_{n-1}t^2D_\lambda) \\ &= v^{-1}\overline{[2]}[D_\lambda - 3E_{n,n}^\theta + 2E_{n,n+2}^\theta + E_{n-1,n}^\theta] + v\overline{[2]}[D_\lambda - 2E_{n,n}^\theta + E_{n,n+2}^\theta + E_{n-1,n+2}^\theta] \\ & \quad + (v^{-\lambda_n-1} + v^{-\lambda_n+1})[D_\lambda - 2E_{n,n}^\theta + E_{n,n+2}^\theta + E_{n-1,n}^\theta] \\ & \quad + (v^{-\lambda_n} + v^{-\lambda_n+2})[D_\lambda - E_{n,n}^\theta + E_{n-1,n+2}^\theta] + (\overline{[\lambda_n]} + v^{-2\lambda_n})[D_\lambda - E_{n,n}^\theta + E_{n-1,n}^\theta]. \end{aligned}$$

After some simplification, we obtain the left hand side of (A.13) as

$$\begin{aligned} & \aleph^i((t^2f_{n-1} + f_{n-1}t^2)D_\lambda) \\ &= (v^{-1}\overline{[2]} + v\overline{[2]})[D_\lambda - 3E_{n,n}^\theta + 2E_{n+2,n}^\theta + E_{n-1,n}^\theta] \\ & \quad + (v^{-\lambda_n+1} + v^{-\lambda_n+3} + v^{-\lambda_n-1} + v^{-\lambda_n+1})[D_\lambda - 2E_{n,n}^\theta + E_{n,n+2}^\theta + E_{n-1,n}^\theta] \\ & \quad + v\overline{[2]}[D_\lambda - 2E_{n,n}^\theta + E_{n,n+2}^\theta + E_{n-1,n+2}^\theta] \\ & \quad + (v^{-\lambda_n} + v^{-\lambda_n+2})[D_\lambda - E_{n,n}^\theta + E_{n-1,n+2}^\theta] \\ & \quad + (\overline{[\lambda_n - 1]} + v^{-2\lambda_n+2} + \overline{[\lambda_n]} + v^{-2\lambda_n})[D_\lambda - E_{n,n}^\theta + E_{n-1,n}^\theta], \end{aligned}$$

For the right hand side of (A.13), we have

$$\begin{aligned} & \aleph^i(tf_{n-1}tD_\lambda) \\ &= \overline{[2]}[D_\lambda - 3E_{n,n}^\theta + E_{n-1,n}^\theta + 2E_{n,n+2}^\theta] + v^{-1}\overline{[\lambda_n - 1]}[D_\lambda - E_{n,n}^\theta + E_{n-1,n}^\theta] \\ & \quad + v^{-\lambda_n+2}[D_\lambda - 2E_{n,n}^\theta + E_{n-1,n}^\theta + E_{n,n+2}^\theta] \\ & \quad + [D_\lambda - 2E_{n,n}^\theta + E_{n-1,n+2}^\theta + E_{n,n+2}^\theta] + v^{-\lambda_n+1}[D_\lambda - E_{n,n}^\theta + E_{n-1,n+2}^\theta] \\ & \quad + v^{-\lambda_n}[D_\lambda - 2E_{n,n}^\theta + E_{n-1,n}^\theta + E_{n,n+2}^\theta] + v^{-2\lambda_n+1}[D_\lambda - E_{n,n}^\theta + E_{n-1,n}^\theta], \end{aligned}$$

and hence,

$$\begin{aligned} & \aleph^i(\overline{[2]}tf_{n-1}t + f_{n-1})D_\lambda) \\ &= \overline{[2]}\overline{[2]}[D_\lambda - 3E_{n,n}^\theta + E_{n-1,n}^\theta + 2E_{n,n+2}^\theta] \\ & \quad + \overline{[2]}(v^{-\lambda_n+2} + v^{-\lambda_n})[D_\lambda - 2E_{n,n}^\theta + E_{n-1,n}^\theta + E_{n,n+2}^\theta] \\ & \quad + \overline{[2]}[D_\lambda - 2E_{n,n}^\theta + E_{n-1,n+2}^\theta + E_{n,n+2}^\theta] + v^{-\lambda_n+1}\overline{[2]}[D_\lambda - E_{n,n}^\theta + E_{n-1,n+2}^\theta] \\ & \quad + \left(\overline{[2]}(v^{-2\lambda_n+1} + v^{-1}\overline{[\lambda_n - 1]}) + 1\right)[D_\lambda - E_{n,n}^\theta + E_{n-1,n}^\theta]. \end{aligned}$$

Now (A.13) follows by comparing the coefficients of equal terms.

The strategy of the proof that \aleph^i is a linear isomorphism is essentially the same as for Theorem 4.7, and let us summarize here. Recall $\dot{\mathbf{U}}^i = \dot{\mathbf{U}}^i(\mathfrak{gl}(N-1))$. The idea is to show that (by restricting to one idempotent summand) monomial bases of $\dot{\mathbf{U}}^i D_{\varepsilon_{n+1}}$ and ${}_{\mathbb{Q}}\mathbf{K}^i[D_{\varepsilon_{n+1}}]$ are parametrized by two sets which are in natural bijection (by passing to $\mathbf{U}(\mathfrak{gl}(N-1))^- D_0$ using [Le02, K14] and ${}_{\mathbb{Q}}\mathbf{K}^-[D_0]$, where ${}_{\mathbb{Q}}\mathbf{K}^-$ is the negative half of ${}_{\mathbb{Q}}\mathbf{K} = \dot{\mathbf{U}}(\mathfrak{gl}(N-1))$ constructed in [BLM]); and then one shows that \aleph^i sends a suitable monomial basis of $\dot{\mathbf{U}}^i D_{\varepsilon_{n+1}}$ to a monomial basis of ${}_{\mathbb{Q}}\mathbf{K}^i[D_{\varepsilon_{n+1}}]$.

The main difference here is that the monomial basis of ${}_{\mathbb{Q}}\mathbf{K}^i$ given in Theorem A.9 needs some adjustment before matching with a monomial basis of $\dot{\mathbf{U}}^i$. Note by (5.4) and Corollary A.14 that the twin products in any monomial appearing in the monomial basis of ${}_{\mathbb{Q}}\mathbf{K}^i$ (see Theorem A.9) is of the form $f_R(\mathbf{t})[D]$ for some polynomial f_R of degree R . We claim that the monomial basis of ${}_{\mathbb{Q}}\mathbf{K}^i$ gives rise to another \mathfrak{l} -monomial basis \mathfrak{M} of ${}_{\mathbb{Q}}\mathbf{K}^i$ when replacing each twin product $f_R(\mathbf{t})[D]$ by its leading term $\mathbf{t}^R[D]$. This claim follows easily by an induction on k for the filtration subspace $Fr^k {}_{\mathbb{Q}}\mathbf{K}^i$. (Here an increasing filtration $\{Fr^k {}_{\mathbb{Q}}\mathbf{K}^i\}_{k \geq 0}$ of ${}_{\mathbb{Q}}\mathbf{K}^i$ is defined by declaring the degrees of the generators of ${}_{\mathbb{Q}}\mathbf{K}^i$ in Corollary A.12, $[D]$, $[D + E_{n,n+2}^\theta]$ (or $\mathbf{t}[D]$), $[D + E_{i,i+1}^\theta]$, $[D + E_{i+1,i}^\theta]$, to be 0, 1, 1, 1.) With the help of the \mathfrak{l} -monomial basis \mathfrak{M} of ${}_{\mathbb{Q}}\mathbf{K}^i$, the same argument in Theorem 4.7 goes through here. \square

The isomorphism $\aleph^i : \dot{\mathbf{U}}^i \rightarrow {}_{\mathbb{Q}}\mathbf{K}^i$ allows one to define an \mathcal{A} -form of $\dot{\mathbf{U}}^i$ as

$${}_{\mathcal{A}}\dot{\mathbf{U}}^i := (\aleph^i)^{-1}(\mathbf{K}^i)$$

such that $\dot{\mathbf{U}}^i = \mathbb{Q}(v) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathbf{U}}^i$ (and ${}_{\mathcal{A}}\dot{\mathbf{U}}^i$ is a free \mathcal{A} -module since so is \mathbf{K}^i). Moreover, the isomorphisms \aleph^i and $\aleph : \dot{\mathbf{U}}^j \rightarrow {}_{\mathbb{Q}}\mathbf{K}^j$ in Theorem 4.7 allow us to transfer the relation between \mathbf{K}^i and \mathbf{K}^j in Theorem A.9 to a relation between $\dot{\mathbf{U}}^i$ and $\dot{\mathbf{U}}^j$. By abuse of notation, we still denote by \mathcal{J} the ideal of $\dot{\mathbf{U}}^j$ which corresponds to $\mathcal{J} \subset \mathbf{K}^j$ (see Lemma A.5).

Proposition A.16. *There is a $\mathbb{Q}(v)$ -algebra embedding $\dot{\mathbf{U}}^i \rightarrow \dot{\mathbf{U}}^j/\mathcal{J}$, which sends the generators D_λ , $e_i D_\lambda$, $f_i D_\lambda$ (for $1 \leq i \leq n-1$ and $\lambda \in {}^i\tilde{\Xi}^{\text{diag}}$) to the generators in the same notation, and sends $tD_\lambda \mapsto f_n e_n D_\lambda - \llbracket \lambda_n - \lambda_{n+1} \rrbracket D_\lambda$; here we recall $\lambda_{n+1} = 1$.*

Proof. By using Proposition 3.7 and keeping in mind $D_{n+1,n+1} = 1$ for $D \in {}^i\tilde{\Xi}^{\text{diag}}$, we have that

$$[D + E_{n,n+1}^\theta] * [D + E_{n+1,n}^\theta] = [D + E_{n,n+2}^\theta] + v^{D_{n,n}-1} \overline{[D_{n,n} + 1]} [D + E_{n,n}^\theta].$$

Setting $D = D_\lambda - E_{n,n}^\theta$ leads to the equivalent formula in the proposition. \square

The bar involution on \mathbf{U}^i (see [BW13]) induces a compatible bar involution on $\dot{\mathbf{U}}^i$, denoted also by $\bar{}$, which fixes all the generators D_λ , tD_λ , $e_i D_\lambda$, $f_i D_\lambda$. The isomorphism \aleph^i intertwines the bar involutions on $\dot{\mathbf{U}}^i$ and on ${}_{\mathbb{Q}}\mathbf{K}^i$, i.e., $\aleph^i(\bar{u}) = \overline{\aleph^i(u)}$, for $u \in \mathbf{U}^i$. We define an \mathcal{A} -subalgebra of $\dot{\mathbf{U}}^i$ by ${}_{\mathcal{A}}\dot{\mathbf{U}}^i := (\aleph^i)^{-1}(\mathbf{K}^i)$ so that $\mathbb{Q}(v) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathbf{U}}^i = \dot{\mathbf{U}}^i$. By definition ${}_{\mathcal{A}}\dot{\mathbf{U}}^i$ is a free \mathcal{A} -submodule of $\dot{\mathbf{U}}^i$ and it is stable under the bar involution.

The isomorphism $\aleph^i : {}_{\mathcal{A}}\dot{\mathbf{U}}^i \rightarrow \mathbf{K}^i$ allows us to transport the canonical basis for \mathbf{K}^i to a canonical basis for ${}_{\mathcal{A}}\dot{\mathbf{U}}^i$. Introduce the divided powers $e_i^{(r)} = e_i^r / \llbracket r \rrbracket!$ and $f_i^{(r)} = f_i^r / \llbracket r \rrbracket!$, for $r \geq 1$. Then we have

$$\aleph^i(e_i^{(r)} D_\lambda) = [D_\lambda - rE_{i,i}^\theta + rE_{i+1,i}^\theta] \quad \text{and} \quad \aleph^i(f_i^{(r)} D_\lambda) = [D_\lambda - rE_{i+1,i+1}^\theta + rE_{i,i+1}^\theta].$$

A.5. Homomorphism from \mathbf{K}^i to \mathbf{S}^i . By construction we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{K}_1^j & \longrightarrow & \mathbf{K}^j \\ \phi_d|_1 \downarrow & & \downarrow \phi_d \\ \mathbf{S}^i & \longrightarrow & \mathbf{S}^j \end{array}$$

where $\phi_d|_1 : \mathbf{K}_1^j \rightarrow \mathbf{S}^i$ is given by a restriction of ϕ_d . The surjective homomorphism $\phi_d|_1$ factors through the ideal \mathcal{J}_1 , and so we obtain a surjective homomorphism $\phi_d^i : \mathbf{K}^i \rightarrow \mathbf{S}^i$.

The following is a counterpart of Proposition 4.11, which is proved in the same way, now by applying Corollary A.12.

Proposition A.17. *There exists a unique surjective \mathcal{A} -algebra homomorphism $\phi_d^i : \mathbf{K}^i \rightarrow \mathbf{S}^i$ such that for $R \geq 0$, $i \in [1, n-1]$ and $D \in {}^i\tilde{\Xi}^{diag}$,*

$$\begin{aligned}\phi_d^i([D + RE_{n,n+2}^\theta]) &= \begin{cases} [D + RE_{n,n+2}^\theta], & \text{if } D + RE_{n,n+2}^\theta \in {}^i\Xi_d; \\ 0, & \text{otherwise;} \end{cases} \\ \phi_d^i([D + RE_{i,i+1}^\theta]) &= \begin{cases} [D + RE_{i,i+1}^\theta], & \text{if } D + RE_{i,i+1}^\theta \in {}^i\Xi_d; \\ 0, & \text{otherwise;} \end{cases} \\ \phi_d^i([D + RE_{i+1,i}^\theta]) &= \begin{cases} [D + RE_{i+1,i}^\theta], & \text{if } D + RE_{i+1,i}^\theta \in {}^i\Xi_d; \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

There is an algebra embedding $\mathbf{U}^i \rightarrow \mathbf{U}(\mathfrak{gl}(N-1))$ (cf. [BW13, Proposition 2.2]), with convention and notation adjusted in a way similar to the embedding $\mathbf{U}^j \rightarrow \mathbf{U}(\mathfrak{gl}(N))$ in Proposition 4.5. Denote by ${}^i\mathbb{V}$ the natural representation of $\mathbf{U}(\mathfrak{gl}(N-1))$. Then the tensor space ${}^i\mathbb{V}^{\otimes d}$ is naturally a $\mathbf{U}(\mathfrak{gl}(N-1))$ -module, which becomes a \mathbf{U}^i -module by restriction, and hence a $\dot{\mathbf{U}}^i$ -module. The right action of the Iwahori-Hecke algebra \mathbf{H}_{B_d} on \mathbf{T}_d^i is similar to the one on \mathbf{T}_d in Lemma 2.4 (and is actually the same as the one given in Lemma 6.2 below).

The action of the Iwahori-Hecke algebra \mathbf{H}_{B_d} on ${}^i\mathbb{V}^{\otimes d}$ is very similar to its action on $\mathbb{V}^{\otimes d}$ given in (4.11)-(4.12). The $(\dot{\mathbf{U}}^i, \mathbf{H}_{B_d})$ -duality established in [BW13, Theorem 5.4] states that the actions of $\dot{\mathbf{U}}^i$ and \mathbf{H}_{B_d} on ${}^i\mathbb{V}^{\otimes d}$ commute and they form double centralizers. We note that the (algebraic) i Schur duality using the Schur algebra instead of the coideal algebra $\dot{\mathbf{U}}^i$ has appeared in [G97]. Similar to the isomorphism $\Omega : \mathbb{V}^{\otimes d} \rightarrow {}_{\mathbb{Q}}\mathbf{T}_d$ in (4.13), now we have an isomorphism $\Omega^i : {}^i\mathbb{V}^{\otimes d} \rightarrow {}_{\mathbb{Q}}\mathbf{T}_d^i$ given by an analogous formula. As a counterpart of Proposition 4.12, we have the following geometric realization of the $(\dot{\mathbf{U}}^i, \mathbf{H}_{B_d})$ -duality.

Proposition A.18. *We have the following commutative diagram of double centralizing actions under the identification $\Omega^i : {}^i\mathbb{V}^{\otimes d} \rightarrow {}_{\mathbb{Q}}\mathbf{T}_d^i$:*

$$\begin{array}{ccccc} {}_{\mathbb{Q}}\mathbf{K}^i & \circlearrowleft & {}_{\mathbb{Q}}\mathbf{T}_d^i & \circlearrowleft & {}_{\mathbb{Q}}\mathbf{H}_{B_d} \\ \uparrow \mathbb{N}^i & & \uparrow \Omega^i & & \parallel \\ \dot{\mathbf{U}}^i & \circlearrowleft & {}^i\mathbb{V}^{\otimes d} & \circlearrowleft & {}_{\mathbb{Q}}\mathbf{H}_{B_d}\end{array}$$

A.6. Compatibility of canonical bases. In this subsection we first work in the setting of Sections 3-4 for \mathbf{S}^j and \mathbf{K}^j . We shall use the notations with subscripts, $[A]_d$ and $\{A\}_d$, to denote a standard and canonical basis element in \mathbf{S}^j , and as before use $[A]$ and $\{A\}$ to denote a standard and a canonical basis element in \mathbf{K}^j , respectively. Recall from Proposition 4.11 the surjective \mathcal{A} -algebra homomorphism $\phi_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$.

Lemma A.19. *The bar involutions on \mathbf{K}^j (as well as on $\dot{\mathbf{U}}^j$) and \mathbf{S}^j commute with the homomorphism $\phi_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$.*

Proof. The lemma follows by checking on the generators with the help of Proposition 4.11. \square

A type A version of the following lemma appears in [DF14, Lemma 6.4], and the proof below follows the one in [Fu12, Proposition 6.3]. We thank Jie Du and Qiang Fu for clarifying our misunderstanding of their crucial lemma.

Lemma A.20. *Let $A \in \tilde{\Xi}$. Then $\phi_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$ sends*

$$\phi_d([A]) = \begin{cases} [A]_d, & \text{if } A \in \Xi_d; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We define an \mathcal{A} -linear map $\phi'_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$ by sending $[A]$ to $[A]_d$ for $A \in \Xi_d$ and to 0 otherwise. We shall show that $\phi'_d = \phi_d$. Observe that ϕ'_d coincides with ϕ_d given in Proposition 4.11 on the generators of \mathbf{K}^j . So it suffices to show that ϕ'_d is an algebra homomorphism. To that end, by the description of generators for \mathbf{K}^j in Proposition 4.10, it suffices to check that

$$(A.14) \quad \phi'_d([B] \cdot [A]) = \phi'_d([B]) * \phi'_d([A]),$$

for $B = (b_{ij}) = [D + RE_{h,h+1}^\theta]$ or $[D + RE_{h+1,h}^\theta]$ (for $R \geq 0$, $h \in [1, n]$ and $D \in \tilde{\Xi}^{\text{diag}}$) and for all $A = (a_{ij}) \in \tilde{\Xi}$. We can further assume without loss of generality that $\text{co}(B) = \text{ro}(A)$ and $\sum_{i,j} a_{ij} = 2d + 1 = \sum_{i,j} b_{ij}$. We will treat the case for $B = [D + RE_{h,h+1}^\theta]$ in detail. The verification of (A.14) is divided into 3 cases.

(1) Assume both $B, A \in \Xi_d$. The identity (A.14) follows directly from the multiplication formulas in Proposition 3.7 and (4.5)–(4.7).

(2) Assume $A \notin \Xi_d$. Then there exists i_0 such that $a_{i_0, i_0} < 0$. It follows by definition that $\phi'_d([B]) * \phi'_d([A]) = 0$. On the other hand, the matrices $A + \sum_u t_u (E_{hu}^\theta - E_{h+1,u}^\theta)$ in the product $[B] \cdot [A]$ (see (4.5)) have negative (i_0, i_0) -entry with possible exceptions when $i_0 = h$ and $t_{hh} > t_{hh} + a_{hh} \geq 0$. In such exceptional cases, the coefficient of such a matrix is a product with one factor $\begin{bmatrix} a_{hh} + t_h \\ t_h \end{bmatrix}$, which is 0. Therefore $\phi'_d([B] \cdot [A]) = 0$ by definition of ϕ'_d , and (A.14) holds.

(3) Assume $B \notin \Xi_d$. By definition we have $\phi'_d([B]) * \phi'_d([A]) = 0$. By assumption, there exists $i_0 \leq n + 1$ such that $b_{i_0, i_0} < 0$. We separate the proof that $\phi'_d([B] \cdot [A]) = 0$ into 2 subcases (i)–(ii) below. (i) Assume $i_0 \neq h$. Then the i_0 -component of $\text{ro}(B)$ is negative and all the resulting new matrices in the product $[B] \cdot [A]$ (see (4.5)) have $\text{ro}(B)$ as their row vectors and so they all contain some negative entry. Thus $\phi'_d([B] \cdot [A]) = 0$. (ii) Assume $i_0 = h$. Then $\text{co}(B)$ has a negative h th component. Since $\text{ro}(A) = \text{co}(B)$, we have $A \notin \Xi_d$. Hence we are back to Case (2) above, and so $\phi'_d([B] \cdot [A]) = 0$.

Summarizing (1)–(3), we have verified (A.14) for $B = [D + RE_{h,h+1}^\theta]$. The completely analogous remaining case for $B = [D + RE_{h+1,h}^\theta]$ will be skipped. The lemma is proved. \square

The next theorem shows that the canonical bases of \mathbf{K}^j and \mathbf{S}^j are compatible under the homomorphism ϕ_d . (A similar result holds in the type A setting; cf. [SV00] and [DF14]).

Theorem A.21. *The homomorphism $\phi_d^j : \mathbf{K}^j \rightarrow \mathbf{S}^j$ sends canonical basis elements for \mathbf{K}^j to canonical basis elements for \mathbf{S}^j or zero. More precisely, for $A \in \tilde{\Xi}$, we have*

$$\phi_d(\{A\}) = \begin{cases} \{A\}_d, & \text{if } A \in \Xi_d; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Lemma A.19, we have $\overline{\phi_d(\{A\})} = \phi_d(\{A\})$, that is, $\phi_d(\{A\})$ is bar invariant. By Proposition 4.2, we have

$$\{A\} = [A] + \sum_{A' \sqsubset A, A' \in \Xi} \pi_{A',A}[A'], \quad \pi_{A',A} \in v^{-1}\mathbb{Z}[v^{-1}].$$

First assume that $A \in \Xi_d$. Then $\phi_d([A]) = [A]_d$ by Lemma A.20, and we have

$$(A.15) \quad \phi_d(\{A\}) = [A]_d + \sum_{A' \sqsubset A, A' \in \Xi_d} \pi_{A',A}[A']_d.$$

By Lemma 3.8, the (geometric) partial order \leq is stronger than \sqsubseteq , and note that the canonical basis element $\{A\}_d$ is characterized by the bar-invariance and the property that $\{A\}_d \in [A]_d + \sum_{A' \sqsubset A, A' \in \Xi_d} v^{-1}\mathbb{Z}[v^{-1}][A']_d$; compare (3.26)–(3.27). By a comparison with (A.15) above, we must have $\phi_d(\{A\}) = \{A\}_d$.

Now assume that $A \notin \Xi_d$. Then $\phi_d([A]) = 0$ by Lemma A.20, and we have

$$\phi_d(\{A\}) = \sum_{A' \sqsubset A, A' \in \Xi_d} \pi_{A',A}[A']_d,$$

which lies in $\sum_{A'' \in \Xi_d} v^{-1}\mathbb{Z}[v^{-1}]\{A''\}$ by applying an inverse version of (3.26). Since $\phi_d(\{A\})$ is bar invariant, it must be 0. The theorem is proved. \square

By definition and Lemma A.20, $\phi_d : \mathbf{K}^j \rightarrow \mathbf{S}^j$ factors through \mathcal{J} and hence we obtain a homomorphism $\bar{\phi}_d : \mathbf{K}^j/\mathcal{J} \rightarrow \mathbf{S}^j$. Then putting all constructions together we have the following commutative diagram

$$(A.16) \quad \begin{array}{ccc} \mathbf{K}^i & \longrightarrow & \mathbf{K}^j/\mathcal{J} \\ \phi_d^i \downarrow & & \downarrow \bar{\phi}_d \\ \mathbf{S}^i & \longrightarrow & \mathbf{S}^j \end{array}$$

It follows by the above results in §A.1–§A.6 that the standard bases are compatible under the two horizontal homomorphisms and $\bar{\phi}_d$ in (A.16), and hence they are also compatible under ϕ_d^i . In the same vein, we apply Theorem A.21 to conclude the following analogue of Theorem A.21, which can be restated that $\phi_d^i : \dot{\mathbf{U}}^i \rightarrow \mathbf{S}^i$ is a homomorphism of based modules in the sense of [Lu93, Chapter 27].

Proposition A.22. *The homomorphism $\phi_d^i : \dot{\mathbf{U}}^i \rightarrow \mathbf{S}^i$ sends canonical base elements to canonical base elements or zero.*

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